The course gives an introduction to various aspects of the field theory of non-equilibrium many-body systems, a young and rapidly evolving area of research. A modern functional integral formulation opens up the powerful toolbox of quantum field theory to non-equilibrium situations, such as the efficient use of collective degrees of freedom or the renormalization group. We develop the theoretical concepts needed to work in this field, and apply them to concrete and prominent physical situations. Topics include:

- Classical dynamical systems:
  - Applications: rare fluctuations, activation problems, dynamical phase transitions: directed percolation (wetting transition), stochastic surface dynamics (Kardar-Parisi-Zhang equation).

- Quantum dynamical systems:
  - Techniques: Quantum Master equations, Keldysh functional integral.
  - Applications: experimental platforms (exciton polariton systems, microcavity arrays), fate of Kosterlitz-Thouless physics in driven open quantum systems, dynamical symmetries, dissipative quantum state engineering, driven closed (Floquet) quantum systems.

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1 Introduction

1.1 What is non-equilibrium physics?

- disclaimer:

\[
\text{equilibrium/non-equilibrium physics} \cong \text{banana/non-banana food}
\]

(adapted from Fradkin)

- but we can roughly structure into:

  \[
  \text{real time dynamics} \quad \rightarrow \quad \text{Hamiltonian (quantum or classical) the only generator of dynamics (typically)}
  \]

  \[
  \text{non-equilibrium stationary states} \quad \rightarrow \quad \text{exchange of:}
  \begin{align*}
  * & \text{energy} \\
  * & \text{particle number} \\
  * & \text{entropy}
  \end{align*}
  \]

  \[
  \rightarrow \text{Hamiltonian not the only dynamical resource}
  \]

  \[
  \rightarrow \text{often: driven systems, typical conservation laws broken}
  \]

Questions:

- quench problems, initial state dependence

- universal transient phenomena (e.g. turbulence \(\cong\) self-driven system via initial state)

- thermalization (in particular, time scales)

- absence of thermalization ("many-body localization")

\[\Rightarrow\] we will get to know several problems in these classes

Questions:

- nature of stationary state

- universal macroscopic phenomena witnessing microscopic driving conditions (new phases, phase transitions, universality classes)

- natural physical systems or new realistic setups
1.2 Why field theory?

- functional integrals provide natural and modern formulation of problems with many degrees of freedom (continuum)

- advanced toolbox available to perform transition from micro- to macro-physics. In particular, flexible in choosing the physically relevant (collective) degrees of freedom; diagrammatic perturbation theory & the renormalization group

- dynamical functional integrals (MSR, Keldysh) leverage power of equilibrium (quantum) field theory over non-equilibrium situations.

  - we will learn such dynamical field theory techniques in both classical and quantum context

1.3 Invitation: The KPZ equation

(Kardar, Parisi, Zhang, PRL 1986)

- a highly non trivial (not entirely solved) paradigmatic example of non-equilibrium universality (does not smoothly connect to equilibrium)

- given by a Langevin equation (non-linear, stochastic, partial differential equation)

\[
\partial_t h(\vec{x}, t) = \nabla^2 h(\vec{x}, t) + \frac{\lambda}{2} (\nabla h)^2 + \xi(\vec{x}, t) \tag{1}
\]

- time evolution of field variable (1) = deterministic evolution (2) & (3) + fluctuating force (stochastic element) (4)

- (1) = (2): heat diffusion equation

- (1) = (2) + (4): "Edward-Wilkinson dynamics": linear stochastic PDE, exactly solvable
(3): non-linearity

how to deal with it?

* map the problem into a field theory
* use tools familiar from field theory, e.g. perturbation theory and RG

cannot exist under equilibrium conditions (!)

* how to detect whether a system in its stationary state is at thermodynamic equilibrium?

→ dynamical symmetries

• physical context here:

  - $h(\vec{x}, t)$ describes a driven, growing interface defined on $\vec{x}, t$
  
  - fluctuating force:
    
    zero mean: $\langle \xi(\vec{x}, t) \rangle = 0$

    short-range correlated:
    
    $\langle \xi(\vec{x}, t) \xi(\vec{x}', t') \rangle \propto \delta(\vec{x} - \vec{x}')\delta(t - t')$

• applications:

  - fire spreading (drive: oxygen)
  
  - bacterial colony growth (drive: sugar)

• to get some intuition, let’s solve the problem with a computer in one dimension (see movies!). We see:

  - the external force is heavily fluctuating for a specific run of the simulation
  
  - it describes the impact of microscopic degrees of freedom, which cannot be tracked completely, on the observable variable $h(\vec{x}, t)$

  - in turn, $h(\vec{x}, t)$ is fluctuating for a specific run
  
  - averages over many runs (“trajectories”) however behave in a smooth, deterministic way. This holds for the stochastic force:

    $\langle \xi(\vec{x}, t) \rangle = 0, \langle \xi(\vec{x}, t) \xi(\vec{x}', t') \rangle \propto \delta(\vec{x} - \vec{x}')\delta(t - t')$  

    and for the observables, e.g.

    $\langle \xi(\vec{x}, t) \rangle = 0, \langle \xi(\vec{x}, t) \xi(\vec{x}', t') \rangle \propto \delta(\vec{x} - \vec{x}')\delta(t - t')$
\[ \langle h(\vec{x}, t) \rangle \propto t \]
\[ w(t) := \frac{1}{L} \int_0^L dx \langle (h(\vec{x}, t) - \langle h(\vec{x}, t) \rangle)^2 \rangle \propto t^{2\beta} \]

power law growth of spatially integrated variance.

- discussion:
  - deterministic physical laws emerge upon averaging over/ "integrating out" the noise!
  - impact of the non-linearity:
    \[ \lambda \neq 0 \Rightarrow \beta = 1/3 \quad \text{non-equilibrium "KPZ fixed point"} \]
    \[ \lambda = 0 \Rightarrow \beta = 1/4 \quad \text{equilibrium "Edwards-Wilkinson fixed point" (see exercises)} \]
    → not smoothly connected even for infinitesimal non-equilibrium perturbation!
    → the size of $\lambda$ sets a time scale at which the KPZ fixed point is approached, though
    → we head for an understanding of this difference

Outline

I. Classical dynamical systems

A) Few degrees of freedom
   → consider (collective) variable $x(t)$ in stochastic environment
   → techniques:

   Langevin equations \(\Rightarrow\) MSR functional integrals \(\Rightarrow\) Fokker-Planck equation (FPE)

   → applications:
      · thermal activation & optimal path approximation in effective phase space, rare fluctuations
      · statistics of work (Jarzynski, 1997)
      · reaction models (chemical reactions)

B) transition to field theory $x(t) \to \phi(\vec{x}, t)$ with continuum of spatial degrees of freedoms
   → techniques:
· straightforward adaption from above
· diagrammatic perturbation theory, dynamic RG

→ applications:
· dynamical symmetries, equilibrium vs. non-equilibrium stationary states
· near equilibrium: dynamics of second order phase transitions, Halperin-Hohenberg-models
· non-equilibrium: directed percolation, absorbing states; KPZ equation

II. Quantum dynamical systems

→ FPE $\equiv$ evolution of probability density $\equiv$ diagonal entries of density matrix
→ start from general open system density matrix evolution

quantum Langevin equations $\Leftrightarrow$ Keldysh functional integral $\Leftrightarrow$ quantum master equations (many degrees of freedom)

→ techniques/formal developments
* Green’s functions, responses vs. correlations
* semiclassical limit (recover above)
* equilibrium limit

→ applications:
* physical systems: microscopic derivation of Keldysh functional integral for light driven semiconductor heterostructures, exciton-polariton-condensates
* 3D: non-equilibrium criticality
* 2D:
  · connection to KPZ equation
  · fate of Kosterlitz-Thouless phase transition out of equilibrium: non-equilibrium duality, vortex unbinding
* field theory of Floquet systems (periodically driven open and closed systems)
* combining drive and dissipation: ”order by dissipation”
  · dark states in quantum optics and many-body systems
  · bosons: dissipatively induced Bose-Einstein-condensation
  · fermions: dissipative Bardeen-Cooper-Schrieffer pairing, topological order
1.4 Literature

- U. C. Täuber, Critical Dynamics, CUP (2014)
Part I

Classical dynamical systems

A) Few degrees of freedom

2 From Langevin equations to the Martin-Siggia-Rose (MSR) functional integral

2.1 Langevin equation for a damped, noisy oscillator

- driven damped oscillator (external forcing)
  \[ m\ddot{x}(t) + \kappa x(t) + \gamma \dot{x}(t) = F_{\text{ext}}(t) \]
  - mass
  - spring constant
  - friction
  - external forcing

- in the absence of external forcing: stationary state with:
  \[ x(t \to \infty) = 0 \] damped oscillations
  \[ \dot{x}(t \to \infty) = 0 \]
  \[ \Rightarrow \text{kinetic energy } E_{\text{kin}}(t \to \infty) = \frac{1}{2}m\dot{x}^2(t \to \infty) = 0 \]
  \[ \text{potential energy } E_{\text{pot}}(t \to \infty) = \frac{1}{2}mx^2(t \to \infty) = 0 \]

- now imagine we couple the oscillator to a heat bath at temperature \( T \). By equipartition theorem we expect:
  \[ \langle E_{\text{kin}} \rangle = \frac{k_B T}{2} = \langle E_{\text{pot}} \rangle \] (3)
  where \( \langle \cdot \rangle \) denotes thermal averaging. Contradicts above findings!

- Q: can this stationary behavior be implemented in the dynamical equations?
A: (Langevin, 1908) Yes. Describe thermal bath as fluctuating (stochastic) force with properties:

- zero mean: \( \langle \xi(t) \rangle = 0 \) - no force on average
- short-time correlations: \( \langle \xi(t) \xi(t') \rangle = A \delta(t - t') \) determined below \((\leftrightarrow \text{temperature})\)

- Picture: Imagine a (very small/light) particle with coordinate \( x(t) \): The fluctuating force would be exerted by the air molecules around it. We only know that the air is at the temperature \( T \).
- Langevin equation:
  \[
  m \frac{\partial^2 x(t)}{\partial t^2} + \kappa x(t) + \gamma \frac{\partial x(t)}{\partial t} = \xi(t)
  \]

- solution: linear differential equation \( \rightarrow \) Fourier transform:
  - \( x(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} x(\omega) \equiv \int_\omega e^{i\omega t} x(\omega); \quad x(\omega) = \int dt e^{-i\omega t} x(t) \)
    \( x(t) \) real \( \Leftrightarrow \) \( x^*(\omega) = x(-\omega) \)
  - analogously,
    \[ \xi(t) = \int_\omega e^{i\omega t} \xi(\omega); \quad \xi(\omega) = \int_t e^{-i\omega t} \xi(t) \]
    real as well properties:
    \[
    0 = \langle \xi(\omega) \rangle = \left\langle \int_\omega e^{i\omega t} \xi(\omega) \right\rangle = \int_\omega e^{i\omega t} \langle \xi(\omega) \rangle \quad \forall t \Rightarrow \langle \xi(\omega) \rangle = 0 \quad (5)
    \]
    \[
    \langle \xi(\omega) \xi(\omega') \rangle = \left\langle \int_\omega e^{-i\omega t} \xi(t) \int_\omega' e^{-i\omega' t'} \xi(t') \right\rangle = \int_t \int_{t'} e^{-i(\omega + \omega') t} \langle \xi(t) \xi(t') \rangle = A \delta(\omega + \omega') \quad (6)
    \]
  - back to solution: Fourier transform
    \[
    \Rightarrow \left[ -m\omega^2 + \kappa + i\gamma\omega \right] x(\omega) = \xi(\omega) \Rightarrow x(\omega) = \frac{\xi(\omega)}{P(\omega)} \quad (8)
    \]
  - computation of averages using the above fluctuating force/noise correlations:
    (i) averages (linear in the observables)
    \[
    \langle x(t) \rangle = \left\langle \int_\omega e^{-i\omega t} \frac{\xi(\omega)}{P(\omega)} \right\rangle = \int_\omega e^{-i\omega t} \frac{1}{P(\omega)} \langle \xi(\omega) \rangle = 0 \quad (9)
    \]
→ oscillator coordinate is zero average ✓ 
\[
\langle \dot{x}(t) \rangle = \left\langle \int_{\omega} e^{-i\omega t} \frac{(-i\omega)\xi(\omega)}{P(\omega)} \right\rangle = ... = 0
\] (10)

→ vanishing mean velocity (force not biased) ✓

(ii) correlations (second moments) however do have non-zero values. Particularly intuitive correlations are:
\[
\langle E_{\text{pot}}(t) \rangle = \langle \frac{1}{2} k x^2(t) \rangle = \frac{1}{2} k \int_{\omega\omega'} e^{i(\omega+\omega')t} \frac{\langle \xi(\omega)\xi(\omega') \rangle}{P(\omega)P(\omega')} = \frac{1}{2} k A \int_{\omega} \frac{1}{P(\omega)^2} = \frac{A}{4\gamma} \tag{11}
\]
\[
\langle E_{\text{kin}}(t) \rangle = \langle \frac{1}{2} m \dot{x}^2(t) \rangle = \frac{1}{2} m \int_{\omega\omega'} e^{i(\omega+\omega')t} \frac{(i\omega)(i\omega')}{P(\omega)P(\omega')} \langle \xi(\omega)\xi(\omega') \rangle = \frac{1}{2} m A \int_{\omega} \frac{\omega^2}{|P(\omega)|^2} = \frac{A}{4\gamma} \tag{12}
\]

→ Discussion:
* All results above are time independent. Indeed, taking the Fourier transform we have assumed time translation invariance. This means that we are looking for the stationary state. Thus, what we calculated should be viewed as the \( t \to \infty \) limit (where stationarity ensues in a damped linear system)
* \( \frac{\langle E_{\text{pot}} \rangle}{\langle E_{\text{kin}} \rangle} = 1 \) in accord with equipartition. We can use the latter to fix the noise strength \( A \):
\[
\langle E_{\text{pot}} \rangle = \langle E_{\text{kin}} \rangle = \frac{A}{4\gamma} = \frac{k_BT}{2}
\] (13)
\[
\Rightarrow A = 2\gamma T \quad \text{Einstein relation} \quad \text{(recognized in the context of Brownian motion)} \tag{14}
\]

⇒ fundamental insight:

The friction (dissipation) \( \gamma \) and the noise level (fluctuation) \( A \) are not independent, but intrinsically tied to each other! Their relation is particularly rigid in thermodynamic equilibrium. The Einstein relation is an instance of Fluctuation-Dissipation-Relation (FDR) → see later!
* Why do we mainly see the deterministic dynamics/damping in everyday life, but not the direct effects of noise?

\(^1\)Brown was a botanist observing random, never stopping motion of pollen in aqueous solution (1827)
Consider scales: Take $x(t)$ collective (center-of-mass) coordinate of an oscillator of mass $m = 1g$. Compute the height it can climb up an oscillator due to thermal noise at room temperature $T = 300K$:

$$\langle E_{\text{pot}} \rangle = mgh = \frac{k_B T}{2} \quad k_B \approx 1.4 \cdot 10^{-23} \text{m}^2\text{kg/s}^2\text{K} \quad \text{Boltzmann constant}$$

$$\sim h \approx 7 \cdot 10^{-18} \text{m} \quad \text{(atomic radius 0.1nm, visible light wavelength 500 nm)}$$

$\Rightarrow$ macroscopic dynamics is effectively zero temperature dynamics

$\Rightarrow$ In this case, it is sufficient to study the conventional Newtonian dynamics, obtained by applying $\langle . \rangle$ to the equation of motion (eom):

$$m\ddot{x}(t) + k\langle x(t) \rangle + \gamma \dot{x}(t) = 0 \quad (17)$$

and computing observables (like $E_{\text{kin}}, E_{\text{pot}}$) naively, as we did in first place above

$\Rightarrow$ more relevant effects of noise occur in the microworld:

- Brownian motion (see above): pollens in water (1827)
- Johnson-Nyquist-noise in $RLC$ circuits

$$\ddot{Q}(t) + \frac{1}{C}Q(t) + \frac{R}{L}\dot{Q}(t) = \frac{U_{\text{ext}}}{U=R\cdot I=R\overline{Q}} \quad \text{voltage drop} \quad + \quad \xi(t) \quad \text{(J.-N.-noise)} \quad (18)$$

\[\begin{array}{c}
\text{Capacitor} \quad C \\
\text{Resistance} \quad R \\
\end{array}\]

2.2 Preparation: multidimensional & functional Gaussian integrals

- This recap is to prepare for the construction of dynamical functional integrals
- significance of Gaussian integrals:
  - exact evaluation possible, work horse for (quantum) field theories.
- physics: non-interacting problems
- perturbation theory reduces interacting problems to the carrying out of Gaussian integrals

**multidimensional Gaussian integral** for real integration variables: we study the discrete partition function

$$ Z[J] = \int \prod_i dx_i \ e^{\frac{i}{2}x^T P x + i x^T J} = \mathcal{N} e^{-\frac{i}{2}J T P^{-1} J}, \quad Z[0] = 1 $$

- \( x, J \in \mathbb{R}^N, \) \( N \) – dimensional vector \( J \) is a source (cf. statistical mechanics),
  \( \mathcal{N} = (2\pi)^{N/2}(i \det P)^{-1/2} \)
- \( P \) is a complex symmetric invertible \( (N \times N) \)-matrix. Being symmetric holds wlog:

$$ x^T P x = x_i P_{ij} x_j = \frac{1}{2}(x_i P_{ij} x_j + x_j P_{ij} x_i) = x_i \left( \frac{1}{2}(P_{ij} + P_{ji}) \right) x_j = x^T \left[ \frac{1}{2}(P + P^T) \right] x, $$

where we use sum convention notation: eg. \( x^T P x = x_i P_{ij} x_j \)
- for the eigenvalues \( \lambda_i \) of \( P \) we need to require (convergence): \( \text{Im} \lambda_i \geq 0 \ \forall i \)
- Eq. (19) seen upon completion of square:

$$ \frac{1}{2}(x^T P x + J^T x + x^T J) = \frac{1}{2}(x^T + J^T (P^{-1})^T) \left[ P + P^{-1} J \right] \left[ \frac{1}{2}(P + P^T) \right] x = \frac{1}{2} J^T P^{-1} J \quad (21) $$

- field expectation value in the absence of sources (1-point correlator)

$$ \langle x_k \rangle := \frac{1}{\mathcal{N}} \int \prod_i dx_i \left[ x_k \right] e^{\frac{i}{2}x^T P x + i J^T x} \Bigg|_{J=0} = \frac{\partial}{\partial (i J_k)} \mathcal{N} \int \prod_i dx_i \ e^{\frac{i}{2}x^T P x + i J^T x} \Bigg|_{J=0} = \frac{1}{2} \frac{\partial}{\partial (i J_k)} (J_k P^{-1} J_k + J_i P_{ij} \delta_{ij}) \Bigg|_{J=0} = \frac{1}{2} (P_{kj} J_j + J_k P_{ij}^{-1}) \Bigg|_{J=0} = 0 \quad (\text{cf. Gaussian centered around 0}) $$

where from first to second line the identity \( x_k = \frac{\partial}{\partial (i J_k)} (i J_k x_i) = \delta_{ik} x_i = x_k \) was used.
- two-point correlator:

\[ \langle x_k x_j \rangle = \left. \frac{\partial}{\partial (iJ_k)} \frac{\partial}{\partial (iJ_j)} Z[j] \right|_{J=0} \]

\[ = \left. \frac{\partial}{\partial J_k} \left[ -\frac{i}{2} (\delta_{ij} P^{-1}_{il} J_l + J_i P^{-1}_{il} \delta_{lj}) e^{i \frac{1}{2} \delta \mathbf{x}^T \mathbf{P} \mathbf{x} + i \mathbf{J}^T \mathbf{x}} \right] \right|_{J=0} \]

\[ = \left. \frac{i}{2} (P^{-1}_{jl} \delta_{lk} + \delta_{lk} P^{-1}_{lj}) \cdot 1 + O(J^2) \right|_{J=0} \]

\[ = \left. \frac{i}{2} (P^{-1}_{jk} + P^{-1}_{kj}) \right|_{P \cdot P^{-1} \text{symmetry}} \]

- in particular, for

\[ P_{kj} = A \delta_{kj} \Rightarrow P^{-1}_{kj} = \frac{1}{A} \delta_{kj} \quad (\delta_{kj} \delta_{ji} = \delta_{kl}) \]  (24)

- higher order moments/ correlators:

→ odd correlators vanish ((2\(\lambda\) + 1)-point functions)

  · differentiation produces terms at least \(O(J)\), vanishing at \(J = 0\)

→ even correlators (2\(\lambda\)-point functions) factorize into products of 2-point functions (special property of Gaussian distribution)

  · clear: the only independent information in \(Z\) is \(P^{-1}_{ij}\),
  and \(\langle x_l x_k \rangle = i P^{-1}_{lk} C_{lk} \)

  · formula (Wick theorem),

\[ \langle x_{l_1} \cdots x_{l_{2\lambda}} \rangle = \sum_P C_{l_{p_1} l_{p_2}} \cdots C_{l_{p_{2\lambda-1}} l_{p_{2\lambda}}} \]  (25)

where the sum runs over all possible pairings \(P\) of \(\{l_1, ..., l_{2\lambda}\}\) ”contractions”.

⇒ proof: 2\(\lambda\) fold differentiation → see exercise

⇒ example for \(\lambda = 2\) (remember \(C_{lk} = C_{kl}\), i.e. indices commute)

\[ \langle x_1 x_k x_j x_m \rangle = C_{lk} C_{jm} + C_{lj} C_{km} + C_{lm} C_{kj} \]  (26)

⇒ higher orders: via recursion

⇒ for the 2\(\lambda\)-point function, we get \(\frac{(2\lambda-1)!}{2^{\lambda-1}(\lambda-1)!}\) terms.

- **Gaussian functional integral** obtains in the continuum limit (0+1 dimensions):

  - notation: (not specific to Gaussian integrals)
* the vector index becomes the field variable argument:
\[ i = 1, \ldots, N \rightarrow t = t_i, \ldots, t_f \]

Limiting procedure: \[ t_f - t_i = N \cdot \delta_i, \ N \rightarrow \infty, \ \delta_i \rightarrow \infty \]
The scaling dimension of \( t \) is inverse energy, \( t \sim E^{-1} \), denoted by \([t] = -1\)

* the vector becomes the field variable:
\[ x_i \rightarrow x(t) \text{ etc. } [x] = 0 \]

* summation becomes integration:
\[ \Sigma_i \rightarrow \int dt, \ [dt] = -1 \]

* the matrix \( P \) becomes the inverse Green’s function:
\[ P \rightarrow P(t, t'), [P] = -2 \]

often, the structure of \( P(t, t') = \delta(t - t') (\partial_t + \ldots) \). \([\partial_t + \ldots] = +1 \), i.e. an energy.

* multidimensional integration becomes functional integration:
\[ \int \prod_{i=1}^{N} dx_i \rightarrow \int \mathcal{D}x \]

• the partition function becomes:
\[
Z[J] = \int \mathcal{D}x \ e^{\frac{i}{2} \int_{t_i}^{t_f} x(t)^T P(t, t') x(t') + \int_{t_i}^{t_f} x(t) J(t)} = (i \det P)^{-1/2} e^{\frac{i}{2} \int_{t_i}^{t_f} J^T(t) P^{-1}(t, t') J(t')} \]

– ordinary (component-wise) derivatives become “functional” derivatives, with all algebraic rules inherited:
\[ \partial \partial J_k \rightarrow \delta_{ik} \quad \rightarrow \quad \frac{\delta J(t)}{\delta J(t')} = \delta(t - t') \]

→ functional chain rule:
\[ \left. \frac{\partial e^{ix_i J_i}}{\partial J_k} \right|_{J=0} = \left. \frac{i \partial (x_i J_i)}{\partial J_i} e^{ix_i J_i} \right|_{J=0} = ix_i \delta_{ik} = ix_k \]
\[ \left. \frac{\delta J(t)}{\delta J(t')} \right|_{J=0} = \left. \frac{i \delta [\int_{t_i}^{t_f} x(t) J(t)]}{\delta J(t')} \right|_{J=0} = i \int_{t_i}^{t_f} x(t) \delta(t - t') = ix(t') \]

→ in particular, for:
\[ P(t, t') = A \delta(t - t') \Rightarrow P^{-1}(t, t') = \frac{1}{A} \delta(t - t') \]

\[ \left( \int_{t''}^{t'''} \delta(t'' - t') \delta(t''' - t) = \delta(t - t') \right) \]

2.3 From Langevin to MSR

(Martin, Siggia, Rose 1973; De Dominicis, 1976; Janssen 1976)

• goal: map stochastic differential equation into dynamical functional integral
(in 0+1 dimensions, higher dimensional generalizations possible: partial stoch. differential eq. \( \leftrightarrow \) (d+1)-dimensional dynamical field theory) remarkable equivalence!

- we study the following generalization of the above Langevin eq.:

\[
\dot{x} = A(x) + b(x)\xi(t)
\]

(29)

- relation to the above case:

- \( A(x) = \frac{\partial V(x)}{\partial x} \) for \( V(x) = \frac{1}{2}kx^2 \) now: arbitrary non-linear function. For example, in the next chapter we study the problem of thermal activation, with \( V(x) \) shown in Fig.1:

Figure 1: Thermal activation

- \( b(x) = 1: \text{"additive noise"} \)
now: arbitrary, noise depends on the collective coordinate \( \rightarrow \text{"multiplicative noise"} \). Example: \( x = \text{density} \); at zero density, there is no noise (absorbing state in directed percolation)

- no second order time derivative: "overdamped limit". This holds when \( \gamma \gg \Omega \sim \frac{\partial^2 V}{\partial x^2} \) typical classical frequency.

- the MSR construction:

- lesson from above: compute expectation value of observable \( Y[x] \) (e.g. \( E_{\text{pot}} = \frac{1}{2}kx^2, E_{\text{kin}} = \frac{1}{2}mx^2 \)) by

(i) inserting for \( x \) the solution \( x_\xi \) of the stochastic equation for noise realization \( \xi \)

(ii) averaging over all noise realizations.

\( \triangleright \) We now formalize the steps in reverse order
– ad (ii): the noise correlators are generated by drawing $\xi$ from a Gaussian distribution:

$$\langle \xi(t) \rangle = 0 \quad \& \quad \langle \xi(t)\xi(t') \rangle = 2\gamma k_B T \delta(t-t') \Rightarrow P(t,t') = \frac{1}{2\gamma k_B T} \delta(t-t')$$  \hfill (30)

Using Eqs. (22),(23) in the continuum limit, we find the associated Gaussian probability distribution:

$$P[\xi] = \frac{1}{N} \exp \left[ -\frac{1}{2} \int_{t,t'} \xi(t)P(t,t')\xi(t') \right] = \frac{1}{N} \exp \left[ -\frac{1}{4\gamma k_B T} \int_t \xi^2(t) \right] \qquad (31)$$

with normalization $N = \int \mathcal{D} \xi \exp \left[ -\frac{1}{2} \int_{t,t'} \xi(t)P(t,t')\xi(t') \right] \Rightarrow \int \mathcal{D} \xi \ P[\xi] = 1$

– ad (i): we use the following $\delta$-functional trick for any observable $Y[x]$:

$$\langle Y[x] \rangle := \int \mathcal{D} \xi \ P[\xi] \ Y[x_{\xi}] = \int \mathcal{D} \xi \int \mathcal{D} x \ P[\xi] Y[x]\delta(x-x_{\xi})$$  \hfill (32)

$$= \int \mathcal{D} \xi \int \mathcal{D} x \ P[\xi] Y[x]\underbrace{J[\sigma]^{-1}[\sigma]}_{\delta(\sigma[x_{\xi}] )} \delta(x-x_{\xi}) \qquad (33)$$

$$= \int \mathcal{D} \xi \int \mathcal{D} x \ P[\xi] Y[x]J[\sigma] \delta(\sigma[x_{\xi}]) \qquad (34)$$

where:

* **key point:** $\sigma$ encodes the full stochastic EoM:

$$O[x] := \partial_t x - A(x) - b(x)\xi \quad \Rightarrow \quad O[x_{\xi}] = 0$$ \hfill (35)

for $x_{\xi}$ solution of EoM, i.e. it has the same zero as $x-x_{\xi}$ (assuming uniqueness).

* The above procedure introduces the **Jacobian** $J[O] := |\det \hat{J}|$ with $\hat{J} := \frac{\partial O}{\partial x}$ describing change of variables. (cf. multidim. $\delta$ function: eg. with label for $\vec{f}, \vec{r}$ - N-dim. real vectors, with $\vec{f}(\vec{r}_0), \vec{r}_0$ being unique zero. Jacobian: $J[\vec{f}] = |\det(\frac{\partial \vec{f}}{\partial x})|$.) We are now going to show that $J = 1$.

→ To this end, we give $J$ a concrete meaning by going back to the discretized version (functional integral $\rightarrow$ multidimensional integral). In this way, we can also specify a time ordering prescription:

- discretization: $x_j = x(t_j) ; \xi_j = \xi(t_j) , j = 1, ..., N$
· Taylor expansion of $O(x)$ (sum convention):

$$O = O_j^{(0)} + O_j^{(1)}x_l + O_j^{(2)}x_lx_m + ...$$

$$\hat{J}_{jk} = \frac{\partial \sigma_j}{\partial x_k} = O_j^{(1)} + \frac{1}{2} \left( O_j^{(2)} + O_j^{(2)} \right) x_m + ...$$

(36) \hspace{1cm} (37)

· to show $J = |\det \hat{J}| = 1$, choose retarded regularization (also known as "Itô regularization"; analogous to normal ordering in quantum mechanics)

$$O_j = x_j - x_{j-1} - \delta_t [A(x_{j-1}) + b(x_{j-1})\xi_{j-1}]; \quad \delta_t = \frac{t_f - t_i}{N} \text{ time step.}$$

(38)

Meaning of retarded: right hand side (rhs) of Eq.(29) is preceding at time $t_{j-1}$.

$\Rightarrow \hat{J}_{jj} = 1; \quad \hat{J}_{j,j-1} = -1 - \delta_t [A'(x_{j-1}) + b'(x_{j-1})\xi_{j-1}]; \quad J_{kl} = 0 \text{ else.}$ (39)

$\Rightarrow \text{the matrix is lower triangular with unit diagonal.}$ (40)

$\Rightarrow J = |\det \hat{J}| = \Pi_{j=1}^N 1 = 1.$

– summary of the preceding steps: Using $J[O] = 1$, we found for any observable $Y[x]$ (discrete version)

$$\langle Y[x] \rangle = \int \Pi d\xi_j \int_j \Pi dx_j \ P[\xi] Y[x] \delta(x_j - x_{j-1} - \delta_t [A(x_{j-1}) + b(x_{j-1})\xi_{j-1}])$$

(41)

– now we use the Fourier decomposition of the $\delta$-functional (cf. discrete case:

$\delta(\vec{r} - \vec{x}) = \int_i \Pi dx_i \ e^{-i \vec{k} \cdot (\vec{r} - \vec{x})}$):

$$\langle Y[x] \rangle = \int \Pi d\xi_j \int_j \Pi dx_j \int_j \Pi d\hat{x}_j \ P[\xi] Y[x] e^{-i \sum_{j=1}^N \hat{x}_j [x_j - x_{j-1} - \delta_t (A(x_{j-1}) + b(x_{j-1})\xi_{j-1})]}$$

(42)
we take the continuum limit $\delta t \to 0$, $N \to \infty$, $\delta t N \to t_f$ (initial time $t_i = 0$)

$$\sum_{j=1}^{N} \frac{\delta t}{\delta t} \left[ \frac{1}{\delta t} (x_j - x_{j-1}) - A(x_{j-1}) \right] = A(x_j) + O(\delta t)$$

$$\int_{t_i}^{t_f} dt \int_{\tilde{x}(t)} dx (x_j - x_{j-1})$$

where "$\partial_t^{Rv}$" reminds of the retarded (Itô) regularization. This yields

$$\langle Y[x] \rangle = \int D\xi \int Dx \int D\tilde{x} \ Y[x] e^{-\frac{i}{4\gamma T} \int_{t_i}^{t_f} \xi^2(t) e^{-i \int_{t_i}^{t_f} \tilde{x}(t) \partial_t^{Rv} (x(t)) - (A(x(t)) + b(x(t))\xi(t))}}$$

finally, we average over the noise; this can be done explicitly due to its Gaussian nature. We use the identity:

$$\frac{1}{4\gamma T} \xi^2 + i \xi \tilde{x} b(x) = -\frac{1}{4\gamma T} (\xi - \frac{i}{2} 4\gamma T \tilde{x} b(x))^2 - 2\gamma T b^2(x) \tilde{x}^2$$

$$\langle Y[x] \rangle = \int Dx \int D\tilde{x} \exp \left( \int_{t_i}^{t_f} dt \left[ -i \tilde{x} (\partial_t^{Rv} x(t) - A(x(t))) - 2\gamma T D(x) \tilde{x}^2 \right] \right)$$

with $D(x) := b^2(x) > 0$; this positivity ensures the convergence of the MSR functional integral.

• MSR partition function

– this derivation was for arbitrary observables $Y[x]$;

– in particular, from Eq.(33) we notice for $Y[x] = 1$: $\langle Y[x] \rangle = 1$.

– a typical $Y[x]$ will be made of monomials $Y[x] = x(t_1) \cdots x(t_n)$; i.e. an n-point correlation function in time. As in statistical mechanics or field theory, this can be generated by introducing sources $\int_{t_i}^{t_f} \tilde{j}(t)x(t)$, and taking successive functional derivatives $\frac{\delta}{\delta \tilde{j}(t_i)}$ to generate the desired monomials. Note that the MSR action depends not only on $x$, but also on $\tilde{x}$. For aesthetic reasons (the meaning and notation will become clear later), we also introduce sources $\int_{t_i}^{t_f} j(t)\tilde{x}(t)$, so to be able to generate correlators $\langle \tilde{x}(t_1) \cdots \tilde{x}(t_m)x(t_{m+1}) \cdots x(t_n) \rangle$. The final MSR partition
function thus reads:

\[
Z[j, \tilde{j}] = \int \mathcal{D}x \mathcal{D}\tilde{x} \ e^{i \left( \frac{S_{\text{MSR}}[x, \tilde{x}]}{2} + \int_{t_i}^{t_f} dt \langle jx + \tilde{j}\tilde{x} \rangle \right)}, \ Z[0, 0] = 1
\]  

(48)

with

\[
iS_{\text{MSR}}[x, \tilde{x}] = \int_{t_i}^{t_f} dt \left[ -i\tilde{x}(\partial_t R x(t) - A(x(t))) - 2\gamma TD(x)\tilde{x}^2 \right]
\]  

(49)

• Discussion

– there is an equivalence of the stationary state of stochastic differential equations and the MSR functional integral. We have shown ”⇒”. For ”⇐”, see exercises. Noise will be introduced naturally in this discussion.

– stationarity: When only interested in stationary correlation functions, eg. \( \langle x(t)x(t') \rangle = f(t - t') \) for arbitrary \( t, t' \), we should choose the time interval \([t_i, t_f] \to [-\infty, \infty] \) for the MSR integral. Picture: (Figure to be inserted!) correlation functions \( \cong ”\text{piercing}” \) of time string with field insertions.

– dynamics: MSR integrand \( e^{iS_{\text{MSR}}} \cong \) probability distribution; time evolution of this distribution by taking \( t_{\text{fin}} < \infty \) and studying local change \( \to \text{Fokker-Planck-equation} \), see below. (importance: initial value problem)

– functional integral fundamentally involves two integration variables (\( \neq \) equilibrium functional integrals!). Interpretation:

  * \( x \) - deterministic (in the sense of \( \langle x \rangle \)) observable, collective coordinate, as in Langevin equation.

  * \( \tilde{x} \) - ”noise” variable (replaces \( \xi \) in stochastic differential equation)

fundamental formal difference between \( x \) and \( \tilde{x} \):

  * \( x \) - appears to arbitrary order in general \( \to \) nonlinear/interacting field theory in 0+1 dimensions.

  * \( \tilde{x} \) appears to quadratic order only (reflecting Gaussian noise). Could be integrated out \( \to \text{Onsager-Machlup-functional} \) (not very useful though). This is characteristic of classical systems. In contrast, in a Keldysh quantum dynamical integral, \( x, \tilde{x} \) appearing symmetrically to the same order.

– use of the two formulations:
* Langevin:
  · initial value problems, time evolution
  · numerical solution

* MSR:
  · boundary value/stationary problems
  · structural/semianalytic understanding
3 Thermal activation: Rare fluctuations and optimal path approximation

Goals:

- we will see the effects of a noise field $\tilde{x}$ in a systematic extension of Newton’s deterministic (damping only) equations
- we will get to know an elegant (but approximate) Hamiltonian (phase space) formulation, allowing for structural insights

3.1 Hamiltonian formulation

- We consider overdamped motion $\partial_t x = -V'(x) + \xi$

- deterministic problem (setting $\xi = 0$ as appropriate in the limit $T = 0$): $x$ settles at a (potentially metastable) minimum

- noisy problem: fluctuating force should allow for an escape ("classical tunneling") out of the metastable minimum.
  $\rightarrow$ we will quantify the escape rate now.

![Figure 2: "Classical" tunneling](image)

- consider the equation of motion of the MSR action (notation: $S_{\text{MSR}} \to S$) from the
principle of least action:

\[ 0 = \frac{\delta S}{\delta \tilde{x}(t)} = \delta \tilde{x}(t) \int_{t^\prime} \left[ \tilde{x}(t^\prime) \left( \partial_{t^\prime} \tilde{x}(t^\prime) - A(x) \right) + i2\gamma TD(x) \tilde{x}^2(t^\prime) \right] \]

\[ = \int_{t^\prime} \left[ -\delta(t - t^\prime) \left( \partial_{t^\prime} \tilde{x}(t^\prime) - A(x) \right) + i4\gamma TD(x) \tilde{x}(t^\prime) \delta(t - t^\prime) \right] \]

\[ = -\dot{x}(t) + A(x) + i4\gamma TD(x) \tilde{x}(t) \]

\[ \Rightarrow \dot{x}(t) = A(x) + i4\gamma TD(x) \tilde{x}(t) \] (50)

\[ 0 = \frac{\delta S}{\delta \tilde{x}(t)} = ... = \dot{\tilde{x}} + A'(x) \tilde{x} + i4\gamma TD'(x) \tilde{x}^2 \]

chain rule

\[ \Rightarrow \dot{\tilde{x}} = -A'(x) \tilde{x} - i4\gamma TD'(x) \tilde{x}^2 \] (52)

where at "..." in Eq.(51) \( \frac{\delta}{\delta x(t)} \int_{t^\prime} \tilde{x} \partial_{t^\prime} x = \frac{\delta}{\delta \tilde{x}(t)} \left[ -\int_{t^\prime} \partial_{t^\prime} \tilde{x} \delta(t - t^\prime) \right] = \int_{t^\prime} \partial_{t^\prime} \tilde{x} \delta(t - t^\prime) = -\partial_t \tilde{x} \) was used.

\section*{Discussion}

- Newtonian/deterministic solution: \( \tilde{x} = 0 \) always provides a solution (noiseless). Even in this case the residual equation \( \dot{x} = A(x) \) still remains to be solved.

  \( \rightarrow \) but there may be other solutions as well!

- \( A(x), D(x) \) real \( \Rightarrow \) \( \tilde{x} \) is purely imaginary (at the stationary points of \( S \)) \( \rightarrow \) simplify notation: \( \tilde{x} := \frac{p}{\gamma^2}; \) \( p(t) \) real on stationary trajectory. Note: This is just one specific configuration of the action, which can be pulled out of the functional integral by a shift of integration variables. There is thus no problem with the convergence of functional integration.

- Key point: The EoM aquire a \textbf{Hamiltonian structure} (phase space variables \( p, x \)):

\[ \dot{x} = \partial_p H(p, x); \ \dot{p} = -\partial_x H(p, x) \quad \text{for} \quad H = pA(x) + p^2 \bar{D}(x) \quad ; \quad \bar{D}(x) := \gamma TD(x) \] (53)
• Note: $p$ is not a physical momentum, cf. the unusual structure of the kinetic term $\propto p!$
Phase portraits (lines of constant energy in phase space) look very differently from those familiar from classical mechanics (e.g. harmonic oscillator: closed elliptic orbits, no fixed point/stationary state).

• nevertheless, the discovery of a Hamiltonian structure allows us to use all properties and intuition from classical phase space dynamics. In particular, recall (many coordinates $\vec{r}, \vec{p}$)

- energy conservation:

\[
\frac{d}{dt} H(\vec{r}, \vec{p}) = \sum_i \left( \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial x_i} \dot{x}_i \right) = \sum_i \left( \dot{x}_i \dot{p}_i - \dot{p}_i \dot{x}_i \right) = 0 \quad (54)
\]

- Liouville theorem:

phase space velocity $\vec{v} = \left( \begin{array}{c} \dot{\vec{r}} \\ \dot{\vec{p}} \end{array} \right)$

\[
\text{div} \, \vec{v} \equiv \sum_i \left( \frac{\partial}{\partial x_i} \dot{x}_i + \frac{\partial}{\partial p_i} \dot{p}_i \right) = \sum_i \left( \frac{\partial^2 H}{\partial x_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial x_i} \right) = 0 \quad (55)
\]

i.e. there are no "sources" in phase space. The Liouville theorem has an interesting consequence for the structure of fixed points (stationary solutions of the EoMs). To this end, consider a fixed point (2D)

\[
\vec{v}_* = \left( \begin{array}{c} v_x \\ v_p \end{array} \right)_* = \left( \begin{array}{c} \dot{x} \\ \dot{p} \end{array} \right)_* = 0 \quad (56)
\]

Figure 3: Fixed points in 2D
Now recall the Gauss theorem projected to 2D:

\[
\text{div} \mathbf{v}(x^*, p^*) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{\partial(\Delta V)} d\mathbf{s} \cdot \mathbf{v} = 0
\]  

(57)

where \( \Delta V \) is the volume centered around \((x^*, p^*)\) and \(\partial(\Delta V)\) its boundary.

⇒ the fixed points are hyperbolic: the number of ”in” arrows must equal the number of ”out” arrows.

- the MSR action reads:

\[
iS[x, p] = -\int_t \left[ p\dot{x} - H(p, x) \right]
\]

(58)

- evaluated at a physical trajectory (optimal path, solution of the EoM’s), by energy conservation we have \(H(p, x)|_{\text{physical}} = \text{constant}\)

- the stationary weight of this path is \(e^{iS}|_{\text{physical solution}}\)

### 3.2 Optimal path approximation and phase portraits

- A reasonable physical picture for the activation problem obtains within the **optimal path approximation (OPA)**: It consists in taking into account only the optimal path in the evaluation of the partition function, and neglecting fluctuations:

\[
Z = \int Dx \, Dp \, e^{iS[x, p]} \overset{\text{shift invariance}}{=} \int D\delta x \, D\delta p \, e^{iS[x_{p.s.}, p_{p.s.}, + \delta x, p_{p.s.}, + \delta p]} = e^{iS[x_{p.s.}, p_{p.s.}]} \int D\delta x \, D\delta p \, \exp \left( \frac{1}{2} \int_{t, t'} (\delta x, \delta p) S^{(2)}[x_{p.s., p_{p.s.}}] \begin{pmatrix} \delta x \\ \delta p \end{pmatrix} + \h.o.t. \right)
\]

(60)

\[
= g[x_{p.s.}, p_{p.s.}] \times e^{iS[x_{p.s.}, p_{p.s.}]} \overset{\text{Gaussian integral}}{\approx} \text{constant} \times e^{iS[x_{p.s.}, p_{p.s.}]}
\]

(61)

- \(x_{p.s.}, p_{p.s.}\) refer to the physical solution, i.e. to the solution of the equation of motion for \(S\) (no fluctuations taken into account). In the second line, we expand in \(\delta x, \delta p\), and use \(\int_t \frac{\delta S[x_{p.s.}, p_{p.s.}]}{\delta x} \delta x = \int_t \frac{\delta S[x_{p.s.}, p_{p.s.}]}{\delta p} \delta p = 0\).

- Validity of optimal path approximation (OPA): low temperatures \(T \to 0\). A hint is given below, where we find \(S|_{p.s.} \sim 1/T\), suggesting a steepest descent suppression argument.
for fluctuations away from the optimal path. See exercises for a more precise justification, where a different route towards the OPA will be taken.

• we are interested in a situation near, but not at the deterministic limit ("rare fluctuations"). Therefore we focus on the phase space trajectory/curve:

\[ H = pA(x) + p^2 \bar{D}(x) = 0. \] Indeed:

- \( H(p = 0, x) = 0 \ \forall x \) describes the deterministic limit
- \( H \) conserved, i.e. \( H(p \neq 0, x) = 0 \) near the deterministic limit

• there are two qualitatively different solutions for \( H = 0 \):

  1. \( p = 0 \): noiseless relaxation; "noiseless solution" in the following
  2. \( p = -\frac{A(x)}{\bar{D}(x)} \): noise/fluctuations are important; "noisy solution" in the following

check: both choices solve the EoM’s:

\[ \dot{x} = A(x) + 2p\bar{D}(x); \quad \dot{p} = -p\partial_x A(x) - p^2 \partial_x \bar{D}(x) \] (62)

ad 1. \( \Rightarrow \dot{x} = A(x) \) remains to be solved; \( \dot{p} = 0 \) consistent
ad 2. insert \( p = -\frac{A}{\bar{D}} \) into second part of Eq. (62):

\[
\Rightarrow \frac{d}{dt} \left( \frac{A}{\bar{D}} \right) = -\frac{\dot{x}\partial_x A}{\bar{D}} + \frac{\dot{x}\partial_x D \cdot A}{D^2}
\]

\[
\overset{(62)}{=} -\left( \frac{A}{\bar{D}} \right) \partial_x A - \left( \frac{A}{\bar{D}} \right)^2 \partial_x D = \frac{A}{\bar{D}} \partial_x A - A \cdot A \cdot \frac{\partial_x D}{D^2}
\]

where in the last equality \( \dot{x} = A + 2 \left( -\frac{A}{\bar{D}} \right) D = -A(x) \) was used.

Discussion:

• the coordinate \( (x) \) dynamics in the noisy solution is **time reversed** compared to the noiseless one.

\[
[\text{recall: time reversal takes } x(t) \mapsto x(-t) = x(\tilde{t}) \]

\[
\Rightarrow \frac{d}{dt} x(t) = A(x(t)) \mapsto \frac{d}{dt} x(-t) = A(x(-t)) = -\frac{d}{dt} x(\tilde{t}) = A(x(\tilde{t})) \] (63)

This property holds always in 1-dimensional systems \((x, p)\) and in multidimensional systems (eg. \( \varphi(\vec{r}, t), p_\varphi(\vec{r}, t) \rightarrow \text{field theories} \)) in thermodynamic equilibrium
• noisy and noiseless trajectories intersect for \( A(x) = 0 \) (cf. Eqs. 1. and 2. above)

• example: overdamped motion in potential \( V(x) \) (such that \( A(x) = -\partial_x V(x) \)) with thermal noise:

- Extrema of the potential landscape:
  \[
  A(0) = 0 \text{ metastable minimum} \\
  A(1) = 1 \text{ unstable maximum}
  \]

- Noisy solution: \( p = \frac{V'(x)}{T} \)

- Stability analysis (how to put the arrows/velocities in phase space)
  * noiseless line \( p = 0 \): only relaxation \( \Rightarrow \) attractive near minimum, repulsive near maximum
  * Liouville theorem: no sources \( \Rightarrow \) at \( p = 0 \) attractive/repulsive fixed points in a 2 dimensional phase space, there must be repulsive/attractive directions as well
  * \[ \text{Implication: In the presence of noise/fluctuations, there must exist an escape path from the deterministic minimum (green path)} \]

- These fluctuations are exponentially rare (where the optimal path approximation is valid, i.e. at low temperature):
  * the stationary weight for reaching \( x = 1 \) is \( e^{iS_{\text{esc.}}} \), where \( S_{\text{esc.}} \) is the action accumulated from the trajectory in phase space connecting \( x = p = 0 \) with \( x = 1, p = 0 \).
  * Initial condition \( H = 0 \) and energy conservation
    \[
    \Rightarrow iS_{\text{esc.}} = - \int_{t_i}^{t_f} dt \, \dot{x} \dot{p} = - \frac{1}{T} \int_{t_i}^{t_f} dt \, x \frac{\partial V}{\partial x} = - \frac{1}{T} \int_{x=0}^{x=1} dx \, \frac{\partial V}{\partial x} = - \frac{V(1) - V(0)}{T}
    \]
escape probability $p \sim e^{-\frac{V(1)-V(0)}{T}}$

* The result only depends on the barrier height (not on the precise shape of the potential and is exponentially small in the latter (Boltzmann factor), but finite: thermal activation through noise fluctuations

* $S \sim \frac{1}{T}$ large for $T \to 0$; necessary for validity of optimal path approximation.

### 3.3 Optimal path approximation and Jarzynski relation

- application of MSR: universal relation for measuring free energy differences via the statistics of performed work: **Jarzynski relation** (1997)

\[
\langle e^{-\frac{W}{T}} \rangle = e^{-(F(t_f)-F(t_i))}
\]  

  - lhs: measurable, $W$ is the performed work; non-equilibrium property/protocol
  - rhs: quantity of interest, free energy difference of final/initial states; equilibrium property

- example of a typical setup: determining the free energy of proteins via the following protocol:

  - stretch the protein, i.e. perform work on it, for fixed time $t_f - t_i$
  - average over many realizations of the experiment
  - the remarkable property is the **independence** of how the precise work protocol is (eg. adiabatic vs. nonadiabatic, at least to the extent that the model description is still valid!)
  - explained by an integrability condition of the MSR action in optimal path approximation
  - (In the protein example, one may be interested in the free energy difference as a function of stretching elongation. So for each elongation, one will just stretch it
many times at the same stretch velocity for the same amount of time.)

- Simplified theoretical model: consider Langevin equation for a collective degree of freedom with time dependent potential

\[
\dot{x} = -\partial_x V(x, t) + \xi(t),
\]

where the explicit time dependence be restricted to the interval \(t_i < t < t_f\)

- the externally imprinted time dependence of \(V(x, t)\) performs work, e.g. a piston compressing a gas, or changing the curvature of a harmonic trapping potential

\[
W[x] = \int_{t_i}^{t_f} dt \partial_t V(x(t), t)
\]

\[
\Rightarrow W \text{ is a functional of the stochastic trajectory } x(t), \text{ and thereby a random quantity, on which only statistical information can be obtained}
\]

- we will derive a universal (path independent, initial and final state dependent) relation for the deterministic observable

\[
\langle e^{-\frac{W}{T}} \rangle = \int D\tilde{x} D\tilde{\tilde{x}} e^{\int [-2i\tilde{x}(\dot{x} + \partial_x V(x, t)) - 4T\tilde{x}^2] - \frac{W[x]}{T}}
\]

\[
\Rightarrow \int D\tilde{x} D\tilde{\tilde{x}} e^{\int [-2i\tilde{x}(\dot{x} - \partial_x V(x, t)) - 4T\tilde{x}^2 - \frac{1}{T} \partial_t V(x, t)]}
\]

- to this end, we argue in OPA and focus on a single action configuration as above. We introduce \(\tilde{x} = \frac{p}{2T}\), \(iS = -\int (p\dot{x} - H[p, x, t])\), with

\[
H[p, x, t] = -p \partial_x V(x, t) + Tp^2 - \frac{1}{T} \partial_t V(x, t) = H_0 - \frac{1}{T} \partial_x V(x, t)
\]

\[
\text{structure as before}
\]
Equation of motion:

\[
\dot{x} = \frac{\partial H}{\partial p} = -\partial_x V + 2Tp, \quad \dot{p} = -\frac{\partial H}{\partial x} = p \partial_x^2 V + \frac{1}{T} \partial_x \partial_t V
\]  \hspace{1cm} (71)

Discussion:

- energy is not conserved due to explicit time dependence
- but remarkably, the form of the above solution still holds for the noisy trajectory:

\[
p = \frac{1}{T} \partial_x V; \quad \dot{x} = +\partial_x V \quad \text{(time reversed compared to noiseless)} \hspace{1cm} (72)
\]

check:

\[
\dot{p} = \frac{1}{T} \frac{d}{dt} (\partial_x V(x,t)) = \frac{1}{T} [\partial_t x \cdot \partial_x^2 V + \partial_x \partial_t V] = \frac{1}{T} [\partial_x V \cdot \partial_x^2 V + \partial_x \partial_t V] \hspace{1cm} (73)
\]

Note, however, that there is no noiseless solution \( p = 0 \) for this problem

- action associated to this trajectory (least action/optimal path)

\[
iS = -\int_t (p\dot{x} - H) = -\frac{1}{T} \int_t (\partial_x V \cdot \dot{x} + \partial_t V) = -\frac{1}{T} \int_{t_i}^{t_f} dt \left[ \partial_x V \cdot \partial_t V \right] \hspace{1cm} (74)
\]

\[
= \frac{V_i - V_f}{T}; \quad V_\alpha = V(x(t_\alpha), t_\alpha), \ \alpha = i, f
\]

\[\implies\] the least action for this problem depends on the initial and final configurations \( x_i, x_f \) only, but not on the path in between!

- approximating the path integral by the contribution from the least action, we get:

\[
\langle e^{-\frac{W}{T}} \rangle = \text{constant} \times e^{-\frac{V_f - V_i}{T}} \hspace{1cm} (75)
\]

- interpretation: \( e^{-\frac{V_f - V_i}{T}} \) is the relative weight/probability for a particle to move from \( x_i \) to \( x_f \)

- to determine the prefactor, we argue physically [formal proof: below!]

* the initial state was known \( (x_i) \), and the associated weight is given by the
Boltzmann distribution:
\[ \frac{e^{-\frac{V_i}{T}}}{Z(t_i)} \text{ with } Z_{th}(t_\alpha) = \int dx \, e^{-\frac{V(x,t_\alpha)}{T}} \equiv e^{-\frac{F(t_\alpha)}{T}} \]

(Note: \( Z_{th} \) is the familiar thermodynamic equilibrium partition function of a single degree of freedom and must not be confused with the MSR partition function!)

* we do not know the final configuration \( x_f \), so we average:

\[ \langle e^{-\frac{W}{T}} \rangle = \frac{e^{-\frac{V_i}{T}}}{Z_{th}(t_i)} \int dx_f \, e^{-\frac{1}{T}(V_f - V_i)} \]

\[ = \frac{1}{Z_{th}(t_i)} \int dx_f \, e^{-\frac{1}{T}V(x_f,t_f)} = \frac{Z_{th}(t_f)}{Z_{th}(t_i)} = e^{-\frac{(F(t_f) - F(t_i))}{T}} \]

The average work imprinted by the non-equilibrium (time dependent) parameter changes depends on the equilibrium free energies of initial and final states only!

This is independent of the protocol, i.e. how and at which speed (even non-adiabatic) parameter changes are done!
4 From the MSR functional integral to the Fokker-Planck-Equation

- a third equivalent reformulation of classical stochastic problems

- compare concepts:
  1. stochastic: solve for variable $x(t)$ in the presence of noise $\xi(t)$, compute correlation functions by noise averaging
  2. deterministic: compute MSR partition function $Z[j, \tilde{j}]$ by integrating exponential of action $S[x, \tilde{x}]$. Obtain expectation values (n-point correlation functions) via functional derivatives with respect to $j, \tilde{j}$
  3. deterministic: evolve an entire probability distribution $P(x, t)$. In the exercises, the direct relation between 1. and 3. is explored.

- goal: establish the relation of 2. and 3. We anticipate that 3. can be regarded as the time-local (differential) formulation of the time-integrated (integral) formulation of MSR. This will allow us to study initial value problems as well.

4.1 From MSR to FPE (single variable)

- We define a probability distribution by a restricted partition function: Namely the functional integral integrated up to time $t$ in $x$, and averaged over the noise $\tilde{x}$ until time $t$, or in discrete version:

$$P(x_j, t_j) := \int \prod_{i=1}^{j-1} dx_i \prod_{i=1}^j d\tilde{x}_i e^{\frac{1}{2} i \tilde{x}_i (x_i - x_{i-1} - \delta_i A(x_{i-1}) - 4 \delta_i \tilde{x}_i^2 D(x_{i-1}))} P(x_1, t_1)$$

- note integration domain: we do not integrate over the deterministic field ($x$, the collective coordinate) at $t_j$, but we average over noise field at $t_j$: stochastic $\rightarrow$ deterministic.

Pictorially:
- Relation to MSR:

- When interested in stationary state properties, and studying a dissipative system with a unique stationary state reached independently of the initial condition, we can send $t_1 = t_{in} \rightarrow -\infty$. We omit reference to that initial state.

- We then recover the MSR partition function as

$$Z = \lim_{t_j \rightarrow \infty} \int dx_j P(x_j, t_j) = \int \mathcal{D}x \mathcal{D}\tilde{x} e^{iS[x, \tilde{x}]} \quad (79)$$

- We now derive the Fokker-Planck equation as the transition of the probability distribution for the deterministic variable $x$, $P(x, t)$, from time $t_{j-1} \rightarrow t_j$

- To this end, we first show: $P(x_j, t_j)$ has indeed the properties of a probability distribution

1. reality: $P(x_j, t_j)^* = P(x_j, t_j)$ after relabeling the integration variables
   $\tilde{x}_i \rightarrow -\tilde{x}_i \forall i$

2. positive (semi-)definite $P(x_j, t_j) \geq 0$ due to exponential form

3. normalization $Z_{t_j} := \int dx_j P(x_j, t_j) = 1 \forall t_j$ (cf. MSR partition function for $j = \tilde{j} = 0$)

$\Rightarrow$ Interpretation: $P(x, t)$: probability of finding a particle at time $t$ at position $x$,
for arbitrary $\tilde{x}$ (integrated out/averaged over)

- to derive EoM for $P(x, t)$, consider evolution under infinitesimal single time step

$$P(x_j, t_j) = \int dx_{j-1} d\tilde{x}_j \exp \left( -2i\tilde{x}_j (x_j - x_{j-1} - \delta_1 A(x_{j-1})) - 4\delta_1 \tilde{x}_j^2 D(x_{j-1}) \right) P(x_{j-1}, t_{j-1})$$

$$iS_{x,j-1}: \text{MSR propagator} \quad (80)$$
now perform integration over $x_{j-1}, \tilde{x}_j$ explicitly using the smallness of $\delta_t$. To this end, first relabel: $\tilde{x}_j \equiv \tilde{x}, x_{j-1} \equiv x_j - \delta x$ (assumption: continuity of $x$)

$$iS_{j,j-1} = -2i\tilde{x} \left( \delta x - \delta_t A(x - \delta x) - 4\delta_t \tilde{x}^2 \begin{pmatrix} D(x - \delta x) \\ \approx D - \delta_x D' + \frac{1}{2} \delta^2 D'' \end{pmatrix} \right)$$

$$= - (\delta x, \tilde{x}) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \tilde{x} \end{pmatrix} + \delta_t (4D\tilde{x}^2 + 2iA\tilde{x} - 2iA'\tilde{x}\delta x + 4D'\tilde{x}^2\delta x - \frac{1}{2}D''\tilde{x}^2\delta x^2)$$

$$=: iS_{\text{Gaussian}}$$

Similarly, we expand:

$$P(x_{j-1}, t_{j-1}) = P(x_j - \delta x, t_j - \delta_t) \approx P(x_j, t_j) - \delta x P' + \frac{1}{2} \delta x^2 P'' - \delta_t \partial_t P$$

Here we have expanded $A$ to first and $D, P$ to second order. This is necessary to work consistently at linear order in the infinitesimal time step (see below). Expanding $A$ and $D, P$ to higher power in Taylor series yields higher orders in $\delta_t$.

We use the notation

$$\langle . \rangle = \int d\delta x \, d\tilde{x} \, e^{iS_{\text{Gaussian}}( . )}$$
where \( S_{\text{Gaussian}} \) is the quadratic form defined above. Then we have:

\[
\begin{align*}
P(x_j, t_j) &= \langle \{1 + \delta_t[4D\tilde{x}^2 + 2iA\tilde{x} - 2iA'\tilde{x}\delta x + 4D'\tilde{x}\delta x - 2D''\tilde{x}^2\delta x^2]\} \rangle \\
\times \{P(x_j, t_j) - \delta x P' + \frac{1}{2} \delta x^2 P'' - \delta_t \partial_t P\} \rangle \\
&= \langle \{1\} \rangle P(x_j, t_j) + P'\delta_t[4D\langle \tilde{x}^2 \rangle + 2iA\langle \tilde{x}\delta x \rangle - 2iA'\langle \tilde{x}\delta x^2 \rangle + 4D'\langle \tilde{x}^2\delta x \rangle - 2D''\langle \tilde{x}^2\delta x^2 \rangle - P'\langle \delta x \rangle - P'\delta_t[4D\langle \tilde{x}^2 \rangle + 2iA\langle \tilde{x}\delta x \rangle - 2iA'\langle \tilde{x}\delta x^2 \rangle + 4D'\langle \tilde{x}^2\delta x \rangle - 2D''\langle \tilde{x}^2\delta x^2 \rangle + 1/2 P''\langle \delta x^2 \rangle + P''\mathcal{O}(\delta_t^2) - \delta_t \partial_t P \langle \{1\} \rangle + \mathcal{O}(\delta_t) \rangle \\
&= \langle \{1\} \rangle P(x_j, t_j) + P'\delta_t[4D\langle \tilde{x}^2 \rangle + 2iA\langle \tilde{x}\delta x \rangle - 2iA'\langle \tilde{x}\delta x^2 \rangle + 4D'\langle \tilde{x}^2\delta x \rangle - 2D''\langle \tilde{x}^2\delta x^2 \rangle + 1/2 P''\langle \delta x^2 \rangle + P''\mathcal{O}(\delta_t^2) - \delta_t \partial_t P \langle \{1\} \rangle + \mathcal{O}(\delta_t) \rangle \\
&= 0
\end{align*}
\]  

- nonvanishing correlation functions (2 dimensional Gaussian integral)

* quadratic order

\[
\begin{pmatrix}
\langle \delta x \delta x \rangle \\
\langle \tilde{x} \delta x \rangle \\
\langle \tilde{x} \tilde{x} \rangle
\end{pmatrix} = \begin{pmatrix}
0 & i/2 \\
i/2 & 0 \\
-i/2 & 0
\end{pmatrix}^{-1} = \begin{pmatrix}
0 & -i/2 \\
-i/2 & 0
\end{pmatrix} (90)
\]

* quartic order

\[
\langle \tilde{x}\tilde{x}\delta x\delta x \rangle \overset{\text{Wick}}{=} 2\langle \tilde{x}\delta x \rangle^2 + 0 = 2 \left( -\frac{i}{2} \right)^2 = -\frac{1}{2} (91)
\]

- To assess the controlledness of the expansion, we need to study higher order correlators: (i) Odd correlators all vanish. (ii) Going beyond the above described expansions for \( A, D, P \) yields higher than quadratic powers of \( \delta x \) in the correlators, while we never have more than two powers of \( \tilde{x} \) (the full action is quadratic in \( \tilde{x} \)). Then applying Wick yields zero for these as there is necessarily a contraction \( \langle \delta x \delta x \rangle = 0 \).

- This leads to

\[
0 = \delta_t[P(-A' + D'') - P'(A' - 2D') + P''D - \partial_t P] (92)
\]
- rearranging terms, we obtain as final result the Fokker-Planck-Equation (FPE)

\[
\partial_t P = -(PA' - P'A) + PD'' + 2P'D' + P''D = -\partial_x(AP) + \partial_x^2(DP)
\]

(93)

- Discussion

- formal analogy to a Schrödinger equation in real space representation
  
  * write \( \partial_t P = \hat{H} P; \)
  
  \[
  \hat{H} = \hat{H}[\hat{P}, x] = \hat{P} A(x) + \hat{P}^2 D(x)
  \]

(94)

* indeed, \([x, \hat{P}] = 1\) for this choice
* this is precisely the ”quantized” version of the classical Hamiltonian discussed in optimal path approximation!
* normal ordering crucial (order of derivatives acting); all \( \hat{P} \) left of \( x \)
* analogy not complete:
  - imaginary time evolution
  - ”wave function” \( P \) real, describes probability, not amplitude
  - \( \hat{P} \) antihermitian

- conservation of probability:
  
  * FPE has structure of a continuity equation:

\[
\partial_t P + \partial_x J = 0 \quad \text{with} \quad J = AP - \partial_x(DP) \quad \text{- probability current density}
\]

(95)

\[
\Rightarrow \partial_t \int dx \, P(x) = \int dx \, \partial_x J = 0
\]

(96)

* on the ”classical”/optimal path level, \( H(\hat{P}, x) \rightarrow H(P, x) \). Probability conservation is then traced back to \( H(P, x) \sim P \), i.e. \( H(P = 0, x) = 1 \)
* note that in functional integral formulation, this translates into \( S[\hat{x} = 0, x] = 0 \)

-
4.2 Summary and generalization (multi-variable systems)

- we compare:
  i) Langevin formulation: \( \vec{r} = (x_1, ..., x_N)^T; \vec{\xi} = (\xi_1, ..., \xi_N)^T \)

\[
\dot{x}_\alpha = A_\alpha(\vec{r}) + b_{\alpha\beta}(\vec{r})\xi_\beta
\]  (97)

using sum convention, and with \( \langle \xi_\alpha(t)\xi_\beta(t') \rangle = 2\delta_{\alpha\beta}\delta(t - t') \)

ii) functional integral formulation
  + stationary states (initial conditions/boundaries of functional integral irrelevant)

\[
Z = \int D\vec{r}D\tilde{\vec{r}} e^{iS}; \quad S = \int dt \left[ -2\tilde{x}_\alpha(\dot{x}_\alpha - A_\alpha(\vec{r})) + 4i\tilde{x}_\alpha D_{\alpha\beta}(\vec{r})\tilde{x}_\beta \right];
\]  (98)

\[
D_{\alpha\beta}(\vec{r}) = b_{\alpha\gamma}(\vec{r})b_{\gamma\beta}(\vec{r})
\]  (99)

- time evolution:

\[
P(\vec{r}_j, t_j) = \int d\vec{r}_{j-1} \vec{r}_j e^{iS_{j,j-1}[\vec{r},\tilde{\vec{r}}]} P(\vec{r}_{j-1}, t_{j-1})
\]  (100)

with \( S_{j,j-1} \) being the action on time slice \( t_{j-1}, t_j \)

* \( P(\vec{r}_j, t_j) \) can be understood as probability distribution evolved up to \( t_j \) under
  "propagator" specified by the action

* \( \int d\vec{r}_j P(\vec{r}_j, t_j) = 1 \forall t_j \)

iii) Fokker-Planck-formulation

\[
\partial_t P(\vec{r}, t) = -\partial_\alpha [A_\alpha(\vec{r}) - \partial_\beta D_{\alpha\beta}(\vec{r})]P(\vec{r}, t)
\]  (101)

with deterministic "drift" \( A_\alpha(\vec{r}) \) and noisy "diffusion" \( D_{\alpha\beta}(\vec{r}) \)

4.3 Applications

- Application 1: Under which conditions is a dynamical system evolving in thermal equilibrium?

  - Comment: FPE is a versatile tool for answering such structural questions; the
conditions will appear as properties of $A_\alpha$, $D_{\alpha\beta}$; thus the results are useful for testing Langevin equations of the MSR action as well (equivalence!)

- 2 independent conditions:
  
a) additive **noise** with structure $\boxed{D_{\alpha\beta} = T\delta_{\alpha\beta}}$
  
b) the **deterministic** term has **potential form** (is an **exact form** in the language of differential calculus),

\[
A_\alpha(\vec{r}) = -\partial_\alpha V(\vec{r})
\]  

(102)

where $V(\vec{r})$ is a scalar potential function

- we show that this entails a Boltzmann/Gibbs ensemble stationary solution:
  
  * **FPE:**

\[
\partial_t P = \partial_\alpha \left[ A_\alpha(\vec{r}) \right] \partial_\beta \left[ D_{\alpha\beta}(\vec{r}) \right] P \quad \text{second order PDE} \quad (103)
\]

* stationary solution (no current) $J = 0$: first order PDE. Using the above condition: $(\partial_\alpha V) P = -T \partial_\alpha P$

\[
\Rightarrow \: P(\vec{r}) = Z_{th}^{-1} e^{-V(\vec{r})/T} \quad \text{Gibbs ensemble/thermal distribution for arbitrary } Z_{th};
\]

choose it as normalization

\[
Z_{th} = \int d\vec{r} e^{-V(\vec{r})/T} = \int \prod_\alpha dx_\alpha e^{-V(\vec{r})/T} \quad (104)
\]

\[
\Rightarrow \: \int \prod_\alpha dx_\alpha P(\vec{r}) = 1 \quad (105)
\]

Discussion:

- key point: this $Z_{th}$ is the **thermodynamic** partition function, not the MSR partition function $Z$! Appreciate this point.

- above: we found the exponential structure in optimal path **approximation**. Here we see that the prefactor is **exactly** $x$-independent!

- in Schrödinger equation analogy, the solution $P(x)$ provides a **zero mode** for the Hamiltonian $\hat{H}[\hat{P}, x]$

- existence of potential form:
  
  * 1 dimensional systems: always possible
* already 2 dimensional systems: **not** guaranteed. Consider:

\[ A_\alpha = -\partial_\alpha V \iff \epsilon_{\alpha\beta}\partial_\alpha A_\beta = -\epsilon_{\alpha\beta}\partial_\beta V = 0 \quad (106) \]

i.e. the existence of potential \( A_\alpha = -\partial_\alpha V \) is equivalent to the assumption of the absence of **vorticity**

* infinite dimensional systems \( \equiv \) field theories

\[
\begin{align*}
\dot{x}_\alpha(t) &\rightarrow \varphi(\vec{r},t) \quad (107) \\
A_\alpha &\rightarrow A_\varphi(\varphi(\vec{r},t)) \overset{\text{eq.}}{=} -\frac{\delta V}{\delta \varphi(\vec{r},t)} \quad (t \text{ is considered fixed}) \quad (108) \\
D_{\alpha\beta} &\rightarrow D_{\varphi,\varphi'} \overset{\text{eq.}}{=} T\delta(\varphi(\vec{r},t) - \varphi'(\vec{r},t)) \quad (109)
\end{align*}
\]

one implementation of the \( \delta \)-constraint realizing equilibrium is to replace it by \( \delta(\vec{r} - \vec{r}') \).

The existence of a potential form is **not** guaranteed (eg. KPZ equation) for field theories. Example: try to represent \((\nabla \varphi)^2\) as functional derivative of an action \( S[\varphi] \); it should not work.

1. Rigorous derivation of Jarzynski relations (fixing the prefactor)

   - use canonical ”quantization” procedure to upgrade the OPA: \( H(P,x) \rightarrow \hat{H}[\hat{P},\hat{x}] \)
   \[ \Rightarrow \text{FPE: } \partial_t P = \partial_x ((\partial_x V)P + T\partial_x^2 P - \frac{1}{T}\partial_t V \cdot P) \]

   - initial condition: equilibrium condition as above,

   \[
P(x,t_i) = \frac{e^{-V(x,t_i)}}{Z_{th}(t_i)} \quad (110)
\]

   - Ansatz: \( P(x,t) = \frac{e^{-V(x,t)/T}}{Z_{th}(t_i)} \forall t \)

   check (note: in the ansatz, the only \( t \)-dependence is the **explicit** one!):

   \[
   -\frac{\dot{V}}{T} P = \partial_x [V' P] + T\partial_x \left[ -\frac{V'}{T} P \right] - \frac{1}{T}[\dot{V}P] \quad (111)
   \]

   - by construction,

   \[
   \langle e^{-W/T} \rangle = \int dx \ P(x,t_f) = \left[ \frac{dx \ e^{-V(x,t_f)/T}}{Z_{th}(t_i)} \right] = \frac{Z_{th}(t_f)}{Z_{th}(t_i)} = e^{-(F(t_f) - F(t_i))/T}
   \]


B) Classical dynamical Many-Particle Systems

5 Dynamics of Phase Transitions

• includes equilibrium (Halperin-Hohenberg models) and genuine non-equilibrium (KPZ equation, directed percolation problem) dynamical criticality

• generalization of the studies of previous chapter to continuum of spatial degrees of freedom

• new key concept: Renormalization group (RG) for dynamical systems. We assume familiarity with this concept here for classical statistical systems and generalize it to dynamical models. See e.g. QFT II script on the Diehl group homepage for a recap of RG theory.

5.1 Dynamics of second order phase transitions: near equilibrium

5.1.1 Models

• transition to field theories in the spatial continuum

\[ \frac{dx_\alpha(t)}{dt} \rightarrow \varphi(x, t) \equiv \varphi(\vec{r}, t) \]

example: damped chains (1 dimension) or membranes (2 dimensions)

\[ \frac{\partial}{\partial t} x_\alpha(t) = -\frac{\partial V[\vec{r}]}{\partial x_\alpha(t)} + \xi_\alpha(t) \]  \hspace{1cm} (113)

cf. discrete case: \[ \frac{\partial}{\partial t} \varphi(x, t) = -\frac{\delta F[\varphi]}{\delta \varphi(x, t)} + \xi(x, t) \]  \hspace{1cm} (112)

Figure 5: Damping of coupled chain

• generalization of Langevin equation of ”potential form” (i.e. there exist a scalar functional whose variational derivative generates determinstic dynamics)
with a "potential" or "free energy"

\[ F[\varphi] = \int d^d r \left[ \frac{D}{2} (\nabla \varphi)^2 + V(\varphi) \right] \] (114)

remarks:

- sometimes referred to as "Hamiltonian", but we have reserved this term for a different object
- first term: elastic compression stiffness of membrane; second: potential landscape
- noise: short range correlated in space and time,

\[ \langle \xi(\vec{r}, t) \xi(\vec{r}', t') \rangle = 2T \delta(\vec{r} - \vec{r}') \delta(t - t') \] (115)

- existence of potential \( F \) not guaranteed in general; represents thermal equilibrium conditions (see Sec. 4.3)
- Eq. (112) is a partial differential equation in contrast to the ordinary differential equation in the first chapter!

- now use equivalence of Langevin equation and MSR functional integral. This leads to the MSR action

\[ S[\varphi, \tilde{\varphi}] = \int_{t, \vec{r}} \left[-2\tilde{\varphi}(\vec{r}, t)\{\partial_t \varphi(\vec{r}, t) + \frac{\delta F[\varphi]}{\delta \varphi(\vec{r}, t)}\} + 4iT \tilde{\varphi}^2(\vec{r}, t)\right], \quad \frac{\delta F[\varphi]}{\delta \varphi(\vec{r}, t)} = -D \nabla^2 \varphi + V'(\varphi) \] (116)

- typical form of the potential

\[ V = \frac{\delta}{2} \varphi^2 + \frac{g}{4} \varphi^4 + h\varphi \] (117)

where \( \delta \) is the "mass" or "gap" (distance from the phase transition) and \( h \) is the external field (\( j \) previously)

- **Symmetries**: To reproduce the symmetries of a given statistical field theory, we have to transform \( \varphi, \tilde{\varphi} \) alike.

- \( h = 0 \): Ising symmetry; the following \( Z_2 \) transformation leaves the MSR action invariant:

\[ \varphi \rightarrow -\varphi, \tilde{\varphi} \rightarrow -\tilde{\varphi} \] (118)
- $h \neq 0$: external magnetic field, Ising symmetry **explicitly** (≠ spontaneously) broken

- extension to more general symmetries straightforward. E.g. real $N$ component field $\varphi_\alpha, \alpha = 1, ..., N$, global $O(N)$ rotations generated by orthogonal matrices $R, R^T R = 1, \vec{\varphi}'(\vec{r}) = R\vec{\varphi}(\vec{r}), \vec{\tilde{\varphi}}'(\vec{r}) = R\vec{\tilde{\varphi}}(\vec{r})$. An invariant action is constructed from

\[
\rightarrow \text{potential term: } V = V(\vec{\varphi}^T \vec{\varphi}),
\]

\[
\rightarrow \text{dynamic term: } \int dt d^d r \vec{\varphi}^T \partial_t \vec{\varphi};
\]

\[
\rightarrow \text{noise term: } \int dt d^d r \vec{\varphi}^T \vec{\tilde{\varphi}}
\]

e.g. $N = 2$: $SO(2) \cong U(1)$, i.e. Bose Einstein condensation of complex scalar fields

- **stationary state**: we show that the stationary state is described by the standard equilibrium partition function (statistical functional integral):

- To this end, we use the equivalence to FPE for probability functional $P([\varphi], t)$
(functional of $\varphi(\vec{r})$ field configuration at time $t$):

$$\partial_t P([\varphi], t) = \int d\vec{r} \frac{\delta}{\delta \varphi(\vec{r})} \left[ \frac{\delta F[\varphi]}{\delta \varphi(\vec{r})} P([\varphi], t) + T \frac{\delta P([\varphi], t)}{\delta \varphi(\vec{r})} \right] J([\varphi], t)$$

(120)

– cf. matrix notation for discrete multivariate systems with potential form:

$$\partial_t P(\vec{r}, t) = \sum_{\beta} \frac{\partial}{\partial x_\beta} \left[ \frac{\partial V}{\partial x_\beta} P(\vec{r}, t) + T \frac{\partial P(\vec{r}, t)}{\partial x_\beta} \right]$$

(121)

– seek current free distribution describing a Gibbs/thermal distribution for the probability, $J = 0$:

$$\frac{\delta}{\delta \varphi} P([\varphi], t) = -\frac{1}{T} \frac{\delta F}{\delta \varphi} P([\varphi], t) \Rightarrow P([\varphi], t) = \frac{1}{Z_{\text{th}}} e^{-\frac{F[\varphi]}{T}}$$

(122)

for any configuration $\varphi_0$. Requiring $P$ to be normalized determines the thermodynamic partition function $Z_{\text{th}}$:

$$Z_{\text{th}} = \int D\varphi e^{-\frac{F[\varphi]}{T}}$$

(123)

* emergence of the classical partition function of thermal equilibrium
* note the appearance of the functional integral ($\int D\varphi = \int \prod_\vec{r} d\varphi(\vec{r})$), opposed to the Riemann integral for single coordinates, and that $\varphi = \varphi(\vec{r})$ is time independent:

$$\Rightarrow \text{The equilibrium functional integral only involves statics}$$

5.1.2 Phenomenology of universality and phase transitions

• we have constructed stochastic models which drive into equilibrium states

• we consider now the late stages of evolution close to, but not at stationary state (linear response), and the dynamical correlations in the stationary state

• focus: universal aspects of dynamics near a critical point

Qualitative overview:

• statement of universality: diverse microscopic systems behave quantitatively identical at large distances (in space and time) when brought near critical point
• example: Bose Einstein condensate and planar magnet, 2-point correlation function

\[ G(\vec{r}) = \langle \varphi(\vec{r})\varphi(0) \rangle \sim e^{-|\vec{r}|/\xi} \quad d > 2 \]  

\(\xi \sim |T - T_c|^{-\nu}\) diverges at critical point

– critical exponents: mass exponent \(\nu\), anomalous dimension \(\eta\)

– universality: these numbers are identical for critical BEC & planar magnet

• physical picture:

- \(\xi \approx (\text{microscopic length scales}) \Rightarrow \) only non-universal behaviour visible

- fine tune to critical point: \(\xi \to \infty \Rightarrow \) universal algebraic scaling "freed", memory of microscopic details lost

• description: Renormalization group (see below); distinguish:

- free theories: Gaussian fixed point (rational critical exponents)

- interacting theories: "Wilson-Fisher fixed point" (WF-FP) (non-rational critical exponents)

- these fixed points are not smoothly connected in \(d < 4\)

• the loss of memory is not complete: symmetries and dimensionality determine the value of critical exponents and define the universality class; examples:

- O(2)/X − Y universality class: see above

- Ising (\(Z_2\) discrete symmetry) universality class: eg. liquid gas transition carbon dioxide; trapped ions
• To give an idea of the amount of loss of memory: there are 80 stable elements, combining just 3 of them leads to $10^6$ possible compounds; the typical systems contain $10^{23}$ particles. But there are only $O(10)$ known universality classes in nature!

• Relation of critical exponents to the dynamical action: study Ising case ($h = 0$)

$$S[\varphi, \tilde{\varphi}] = \int_{\vec{r},t} -2\tilde{\varphi} \left[ (\partial_t + D\nabla^2)\varphi + V'(\varphi) \right] + 4iT\tilde{\varphi}^2$$ (125)

- neglect time and space dependence of the field ("Mean field") $\varphi(\vec{r}, t) = \varphi_0 \to$ only the potential term contributes, and we distinguish three qualitatively distinct situations:

  - here, $\delta \sim T - T_c$ is the tuning parameter through the phase transition, vanishing at the critical point

  - **observation** (experiments): all observables exhibit power law scaling dependence on $\delta$ as $\delta \to 0$. In particular, we have:

    correlation length: $\xi \propto |\delta|^{-\nu}$ see above: algebraic decay of $\langle \varphi(r, 0)\varphi(0, 0) \rangle$

    correlation time: $\tau \propto |\delta|^{-\nu z} \propto \xi^z$: algebraic decay of $\langle \varphi(0, t)\varphi(0, 0) \rangle$

  - prediction of critical exponents $\nu, z$ (correlation length exponent, dynamical exponent) from Gaussian theory ($g = 0$): simple dimensional analysis

    $[\varphi] = \text{Energy}$
    $[\nabla^2] = \text{Energy} \Rightarrow \text{Energy} \propto \frac{1}{\text{Length}^2}$
    $[\partial_t] = \text{Energy} \Rightarrow \text{Energy} \propto \frac{1}{\text{Time}}$
    $\Rightarrow \xi \propto |\delta|^{-\frac{1}{2}} \Rightarrow \nu = \frac{1}{2}$
    $\tau \propto |\delta|^{-1} \Rightarrow \nu z = 1 \Rightarrow z = 2$

• This prediction is correct for (above "upper critical dimension") but wrong for the physical cases $d < 4$. The exponents are still fully universal, but interactions become important **irrespective** to their **microscopic smallness**, and affect the exponents. The reason for
this is anticipated from perturbation theory

$$\Delta g = g \cdot I \cdot g$$  \hspace{1cm} (126)

Where $I$ is the sum over intermediate states, which turns out to be

$$I \propto \int dq \frac{q^{d-1}}{q^4} = \int d(\log q) q^{d-4}$$  \hspace{1cm} (127)

$\Rightarrow$ this is an infrared (long wavelength) divergence leading to a blowup of perturbation theory in a critical ($\delta \to 0$) system with spatial continuum of degrees of freedom $\to$ control via Renormalization group (RG) (see QFT II).

5.1.3 Renormalization group for dynamical field theories

- new elements:
  
  i) statistical field theory: degrees of freedom equipped with variable $\vec{q}$ (or $\vec{r}$); dynamical field theory: $Q = (\omega, \vec{q})$ (or $\vec{R} = (t, \vec{r})$)
  
  ii) doubling of degrees of freedom $(\varphi, \tilde{\varphi})$: two field types, additional discrete index

- how to deal with them, guided by pragmatic simplicity:
  
  i) we do not regularize frequency shells, but do the integrals (usually simple due to residue theorem) in an unrestricted way. Graphically:

  ![Graphical representation of integral](image)

  This usually works since the momentum shell regularization is sufficient to regularize the IR divergences

  ii) write down the simplest action capturing the qualitative physics of the phase transition, and respects the symmetries. The phase transition is described by a qualitative change of phase space topology (only homogeneous terms correspond to relevant operators)
We exemplify this for the case of Ising symmetry (change variables to $\pi = 2i\tilde{\varphi}$): discrete field transformation $(\pi, \varphi) \to (-\pi, -\varphi)$. We can formulate an Ising symmetric MSR action by

$$H = \int \mathbf{r} \left( \gamma T \pi - \delta \varphi + D \nabla^2 \varphi - g \varphi^3 \right)$$  \hspace{1cm} (128)

$$\Rightarrow iS = \int \mathbf{r}, t \left\{ \pi \left( \gamma \partial_t \varphi - D \nabla^2 + \delta \varphi + g \varphi^3 \right) - \gamma T \pi^2 \right\}$$  \hspace{1cm} (129)

- looks like the simplistic microscopic model; constants are defined phenomenologically here, but their precise values (once nonzero and equipped with the right signs) do not matter for the universal behavior
- higher order terms do not change the qualitative physics. Classified as irrelevant, see below.

• application of the RG program:

1. Dimensional analysis
   - canonical rescaling of space/time and fields:

   $$\mathbf{r} \to b\mathbf{r}, \ t \to b^z t \quad z \text{ measures relative scaling of space and time}$$  \hspace{1cm} (130)

   $$\varphi(\mathbf{r}, t) \to b^x \varphi(b\mathbf{r}, b^z t), \ \pi(\mathbf{r}, t) \to b^x \pi(b\mathbf{r}, b^z t)$$  \hspace{1cm} (131)
canonical rescaling of the 5 coupling constants from the requirement of a dimensionless action (scaling $\sim b^0$):

\[
\begin{align*}
\gamma' &= b^{d+\tilde{\chi}+\chi} \gamma, \\
D' &= b^{d+z-2+\tilde{\chi}+\chi} D, \\
\delta' &= b^{d+z+\chi} \delta \\
g' &= b^{d+z+\chi} g, \\
(\gamma T)' &= b^{d+2+2\tilde{\chi}} (\gamma T)
\end{align*}
\]

$\Rightarrow$ 3 free parameters $z, \chi, \tilde{\chi}$ (cf. a single one in the statistical field theory: $\chi$). To fix these on the level of canonical power counting, require physically dimensionless parameters $\gamma, D, \gamma T$ not to scale (they are coefficients to expressions with dimension energy in the Lagrangian density)

\[
\begin{align*}
\gamma : \quad d + \tilde{\chi} + \chi &= 0, & z = 2 \text{ (cf. diffusion)} \\
D : \quad d + z - 2 + \tilde{\chi} + \chi &= 0 \Rightarrow \chi = \frac{2-d}{2} \\
\gamma T : \quad d + z + 2\tilde{\chi} &= 0 \Rightarrow \tilde{\chi} = -\frac{2+d}{2}
\end{align*}
\]

$\Rightarrow$ splitting of the canonical dimension of noise and deterministic field. NB: These are not exact relations. They select a physically motivated scaling solution of the action. Loop corrections will modify the values of $z, \chi, \tilde{\chi}$.

Discussion:

- Classification of coupling constants ($d_x$: canonical dimension of $x$)

\[
\begin{align*}
d_{\delta} &= 2 \quad \text{relevant} \\
d_{g} &= 4 - d \quad \text{relevant}
\end{align*}
\]

and identical to statistical field theory! $\varphi^2, \varphi^4$ term

- due to splitting of field dimensions $\chi, \tilde{\chi}$: the only non-irrelevant couplings in $d > 2$ are (i) the coefficient of $\pi^2 (d_{\gamma T} = 0)$ and (ii) deterministic terms linear in $\pi$. To see (ii), we compute the dimension of $g^{(n,m)}$ for $n = 2, m \geq 1$:

\[
g^{(n,m)} \int dt \, d^d x \, \pi^n \varphi^m \\
\Rightarrow d_{g}^{(2,m)} = d + 2 - \frac{2}{2}(2 + d) + \frac{m}{2}(2 - d) \\
= m(1 - \frac{d}{2}) < 0 \text{ for } d > 2 \rightarrow \text{irrelevant}
\]
- high order couplings in $\varphi$ are also irrelevant in large enough dimension: consider

$$g^{(1,m)} \int dt \, d^d x \, \pi \varphi^m$$

$$\Rightarrow d_g^{(1,m)} = \frac{d}{2} (1 - m) + (1 + m) \frac{1}{2} = 0 \quad (< 0 \text{ - irrelevant})$$

$$\Rightarrow m_{\text{crit}} = \frac{(\frac{d}{2} + 1)}{(\frac{d}{2} - 1)}$$

- $d = 2$ is lower critical dimension ($m_{\text{crit}} = \infty$) - couplings with arbitrary power of $\varphi$ are not irrelevant. This is also seen from the fact that $d_\chi = 0$ in $d = 2$. Physical consequence: absence of useful power counting: arbitrary powers of the field $\varphi$ contribute equally; importance of topologically nontrivial field configurations (vortices), **Kosterlitz-Thouless transition**

- in $d = 3$, $m_{\text{crit}} = 5$

2. **Inclusion of fluctuations**

- the only relevant nonlinear term in $d \geq 3$ is:

$$g\pi \varphi^3 =$$

- decompose into slow and fast fields,

$$\pi = \pi_s + \pi_f; \quad \varphi = \varphi_s + \varphi_f$$

In particular, we will need the vertex with $2s, 2f$ fields for the fluctuation correction. Their multiplicity is:

$$3g\pi_s \varphi_s \varphi_f^2; \quad 3g\pi_f \varphi_f \varphi_s^2$$

- to compute the relevant Gaussian correlators, first symmetrize the quadratic
part of the action to properly account for the discrete index structure:

\[
iS = \int_{t,\vec{r}} \left\{ \frac{1}{2}(\varphi, \pi) \begin{pmatrix} 0 & -\gamma \partial_t \varphi - D \nabla^2 \varphi + \delta \\ \gamma \partial_t \varphi - D \nabla^2 \varphi + \delta & -2\gamma T \end{pmatrix} \begin{pmatrix} \varphi \\ \pi \end{pmatrix} \right\} + g \pi \varphi^3 \right\} \text{Interaction } S_{int}
\]

(140)

Bring into Fourier space \((Q = (\omega, \vec{q}))\),

\[
iS_G = \frac{1}{2} \int_Q \begin{pmatrix} \varphi(-Q) \\ \pi(-Q) \end{pmatrix} \begin{pmatrix} 0 & -i\gamma \omega + Dq^2 + \delta \\ i\gamma \omega + Dq^2 + \delta & -2\gamma T \end{pmatrix} \begin{pmatrix} \varphi(Q) \\ \pi(Q) \end{pmatrix},
\]

(141)

\[
iS_{int} = \int_{Q_1, \ldots, Q_4} \delta(Q_1 + Q_2 + Q_3 + Q_4) \pi(Q_1) \varphi(Q_2) \varphi(Q_3) \varphi(Q_4).
\]

- The inverse of the matrix \(G^{-1}\) gives the Gaussian correlators/ the bare Green’s function:

\[
G(Q, Q') = \begin{pmatrix} \langle \varphi(Q)\varphi(Q') \rangle & \langle \varphi(Q)\pi(Q') \rangle \\ \langle \pi(Q)\varphi(Q') \rangle & \langle \pi(Q)\pi(Q') \rangle \end{pmatrix} = G(Q)\delta(Q + Q'),
\]

(142)

\[
G(Q) = \begin{pmatrix} G^K(Q) & G^R(Q) \\ G^A(Q) & 0 \end{pmatrix} = \begin{pmatrix} \frac{2\gamma T}{\gamma^2\omega^2 + (Dq^2 + \delta)^2} & \frac{1}{i\gamma \omega + Dq^2 + \delta} \\ \frac{1}{i\gamma \omega + Dq^2 + \delta} & 0 \end{pmatrix}
\]

- nomenclature:

* \(G^{R/A}(Q) = \frac{1}{i\gamma \omega + Dq^2 + \delta}\) **retarded/advanced Green’s function** because \(G^{R/A}\) has poles in the lower/upper half complex plane, thus

\[
\int e^{i\omega(t-t')} G^{R/A}(\omega, \vec{q}) \propto \theta(\pm(t-t'))
\]

(143)

* \(G^K(Q) = \frac{2\gamma T}{\gamma^2\omega^2 + (Dq^2 + \delta)^2} = -G^R(-2\gamma T)G^A\) statistical (later: "Keldysh") Green’s function

* the zero entry \(\langle \pi(Q)\pi(Q') \rangle = 0\) reflects causality of the MSR action, \(S[\pi = 0, \varphi] = 0\) for any \(\varphi\) (exact property)

→ Now we apply the perturbative RG program:
i) mass correction: (external fields $\pi_s$, $\varphi_s$)

$$\Delta \delta = -3g \int \frac{d\omega}{2\pi} \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} G^K(Q)$$

$$= -3g \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{T}{Dq^2 + \delta} \delta \rightarrow 0 \approx -3g \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} T \frac{Dq^2}{(Dq^2)^2} + 3g \cdot \delta \cdot \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} T \frac{Dq^2}{(Dq^2)^2}$$

(144) (145)

Discussion:

- first term:
  * $\delta$-independent, additive contribution to ”bare” $\delta \rightarrow$ shifts critical point,
    absorbed into definition of $\delta$
  * ultraviolet (UV) dominated $\rightarrow$ ”additive renormalization”

- second term:
  * IR dominated, contains critical divergence $\rightarrow$ take into account in RG
  * linear in $\delta$ $\rightarrow$ ”multiplicative renormalization” of $\delta$

- interaction $Q$-independent $\Rightarrow$ no momentum dependence of the self-energy
  $\Rightarrow \gamma$ and $D$ not normalized at this order perturbation theory

ii) vertex correction: $\pi_s\varphi_s^3$
\[
\Delta g = 9g^2 \cdot 4 \int \hat{d}\omega \int_{\Lambda/b}^{\Lambda} \hat{d}^4 q \, G^K(Q) G^R(Q)
\] 
(146)

\[
\delta = 0 \quad \text{leading order}
\]

\[
9g^2 \int_{\Lambda/b}^{\Lambda} \hat{d}^4 q \, \frac{T}{(Dq^2)^2}
\] 
(147)

\(\rightarrow\) also here the integral is IR divergent \(\sim \Lambda^{d-4}\), i.e. it blows up below \(d = 4\).

- Combination: RG equations \(b = 1 + l, \ l \to 0:\)
  \(\partial_l \gamma = \partial_l D = \partial_l (\gamma T) = 0\) zero canonical dimension, no 1-loop fluctuation correction
  \(\partial_l \delta = 2\delta - 3K_d \delta \cdot g\)
  \(\partial_l g = (4 - d)g - 9K_d g^2\)
  
  - here, \(K_d = \frac{T}{D^2} \frac{8_d}{(2\pi)^d} \Lambda^{d-4}, \ S_d = 2\pi^{d/2}\Gamma(d/2)\) (Euler Gamma).
  
  - **NB:** This reproduces exactly the result of the equilibrium calculation for \(N = 1\) (Ising): See exercises. The reason is that \(\delta, g\) are **static** couplings (frequency independent). We have shown in Sec. 5.1 that the equilibrium stationary state coincides with the statistical partition function

- more detailed discussion:

  - scale invariance: \(\Leftrightarrow \partial_l g_i = 0 \ \forall i, \ \{g_i\} \text{ the set of couplings}\)
  
  - applied to the mass term (fine tuned to criticality)
    
    \[
    \partial_l \delta = (2 - 3K_d g)\delta \quad \Rightarrow \quad \delta = \delta_0 e^{(2 - 3K_d g^*)l} = \delta_0 b^{2 - 3K_d g^*}
    \]
    
    - thus, we get the scaling of correlation length and time as:

      \[
      \xi \propto b \propto \delta_0^{-\frac{1}{2-3K_d g^*}} \propto 0^{-\nu}
      \]  
      \(\Rightarrow\)

      \[
      \tau \propto b^2 \propto \delta_0^{-\frac{1}{2-3K_d g^*}} \propto 0^{-\nu_z}
      \]  
      \(\Rightarrow\)

- applied to the coupling term:

  \[
  \partial_l g_s = 0
  \]  
  (150)

  there are two solutions (\(\epsilon = 4 - d\)),

  \(\cdot\)
i) **Wilson Fisher FP**: \( g_s = \frac{\epsilon}{g_K^2} \approx \frac{\epsilon}{g_K^4} \), stable (only one relevant direction corresponding to the physical fine tuning to the critical point)

\[
\nu = \frac{1}{2} \left( 1 - \frac{\epsilon}{6} \right), \quad \nu_z = 1 - \frac{\epsilon}{6}
\]  

(151)

ii) **Gaussian FP** \( g_s = 0 \), unstable, mean field behaviour

\[
\nu = \frac{1}{2}, \quad \nu_z = 1
\]  

(152)

\( \Rightarrow \) non-Gaussian/anomalous scaling of length and time scales due to nontrivial Wilson-Fisher FP for \( g \).

- so far we discussed static and dynamic couplings on equal footing. We will now see that models with the same static exponents can have different dynamic exponents, depending on symmetries. This is the essence of the Halperin-Hohenberg classification, revealing a fine structure in dynamical criticality

### 5.1.4 Halperin-Hohenberg classification

**goals:**

- study one dynamical modification of "Model A" considered above (the most generic model), called "Model B"

- show that all static properties of Model A are reproduced in Model B

- point out differences in dynamical properties between these models

**Model B: dynamics with a conserved order parameter**

- conservation law: additional constraint on \( \varphi(\vec{r},t) \):

\[
\int d^d r \varphi(\vec{r},t) = \varphi(\vec{q} = 0,t) \equiv \text{constant}
\]  

(153)

- example: \( \varphi \) is the particle density in liquid-gas transition, or the order parameter of an (uniaxial) magnet with conserved spin

- implementation of constraint via continuity equation:

\[
\partial_t \varphi(\vec{r},t) = -\nabla \vec{J}; \quad \vec{J} = -\nabla \frac{\delta F}{\delta \varphi(\vec{r},t)} - \vec{\xi}(\vec{r},t)
\]  

(154)

with

\[
\langle \xi_i(\vec{r},t)\xi_j(\vec{r}',t') \rangle = 2T \delta_{ij} \delta(\vec{r} - \vec{r}') \delta(t - t')
\]  

(155)
the gradient nature of the current is dictated by the scalar nature of the lhs and rotation invariance; \( F \) is a polynomial in the fields and its derivatives as above

- MSR action (following the usual construction)

\[
i S[\varphi, \pi] = \int dt \left[ \int d^d r \left( \pi \partial_t \varphi - H[\pi, \varphi] \right) \right],
\]

\[
H(\varphi, \pi) = \int d^d r \left( -\nabla \pi \nabla \frac{\delta F}{\delta \varphi(\vec{r}, t)} + T(\nabla \pi)^2 \right)
\]

- we concentrate on two statements:

  i) The equilibrium state of model B is identical to model A

  - work in optimal path approximation (upgrade via FPE possible)
  - calculate (relative) probability of reaching configuration \( \varphi_0(\vec{r}) \) from \( \varphi = 0 \) through zero energy activation trajectory

\[
\rightarrow 0 = \frac{1}{H} = \int d^d r \left( -\nabla \pi \nabla \left[ \frac{\delta F}{\delta \varphi(\vec{r}, t)} - T \pi \right] \right)
\]

1) relaxation trajectory \( \pi = 0 \):

\[
\partial_t \varphi = \frac{\delta H}{\delta \pi} = \nabla^2 \frac{\delta F}{\delta \varphi} - 2T \nabla^2 \pi \bigg|_{\pi = 0} \nabla^2 \frac{\delta F}{\delta \varphi}
\]

\( \rightarrow \) the relaxation trajectory involves the additional \( \nabla^2 \) operator compared to Model A. I.e. the dynamics sees the difference between Model A and B.

2) activation trajectory \( \pi = \frac{1}{T} \frac{\delta F}{\delta \varphi} \)

**NB:** this equality does not involve the additional \( \nabla^2 \) operator. This is the reason for the equality of the stationary states, see below. The relation is identical to Model A

\[
\Rightarrow \partial_t \varphi = -\nabla^2 \frac{\delta F}{\delta \varphi} \text{ time reversed, sees } \nabla^2
\]
→ action:

\[ S = \int dt \, d^d r \, \pi \partial_t \varphi = \frac{1}{T} \int dt \, \frac{\delta F}{\delta \varphi} \partial_t \varphi = F(\varphi(\vec{r})) - F(\varphi = 0) \]  \hspace{1cm} (161)

→ probability:

\[ P[\varphi_0(\vec{r})] = Z_{th}^{-1} e^{-\frac{F[\varphi_{\varphi}(\vec{r})]}{T}} \quad \forall \varphi \hspace{1cm} (162) \]

\[ \text{with } Z_{th} = \int D\varphi(\vec{r}) e^{-\frac{F(\varphi)}{T}} \] \hspace{1cm} (163)

⇒ all static properties identical to Model A (including static critical exponents!)

It is instructive to verify this statement in perturbation theory:

– consider low energy effective action

\[ S = \int dt \, d^d r \, \left[ (\gamma \pi \partial_t \varphi + \pi \nabla^2 (D\nabla^2 \varphi - \delta \varphi - g\varphi^3)) - \gamma T (\nabla \pi)^2 \right] \] \hspace{1cm} (164)

additional gradients!

– free propagators (note the additional $q^2$ terms)

\[ G^R(Q) = \frac{1}{-i\gamma \omega + q^2(Dq^2 + \delta)} \] \hspace{1cm} (165)

\[ G^K(Q) = \frac{2\gamma T q^2}{(\gamma \omega)^2 + (q^2(Dq^2 + \delta))^2} \] \hspace{1cm} (166)

– vertex:

\[ -g(\nabla^2(\pi_s + \pi_f))(\varphi_s + \varphi_f)^3, \text{ i.e. contribution to RG:} \] \hspace{1cm} (167)

\[ -3g(\nabla^2 \pi_s)\varphi_s^2; \quad -3g(\nabla^2 \pi_f)\varphi_f^2, \] \hspace{1cm} (168)

– diagrams: mass and vertex corrections
\[
\begin{align*}
\ast \\
&= -3g \int d\tilde{\omega} \int_{\Lambda/b}^{\Lambda} \tilde{d}^4 q \langle \varphi(-Q)\varphi(Q) \rangle = -3g \int_{\Lambda/b}^{\Lambda} \tilde{d}^4 q \frac{q^2 T}{q^2 (Dq^2 + \delta)} \\
\rightarrow & \text{ identical to Model A!}
\end{align*}
\]

\[
\begin{align*}
&= 9g^2 \int d\tilde{\omega} \int_{\Lambda/b}^{\Lambda} \tilde{d}^4 q \langle \varphi(-Q)\varphi(Q) \rangle \langle \pi(-Q)\varphi(Q) \rangle \cdot q^2 \\
&= 9g^2 T \int_{\Lambda} \tilde{d}^4 q q^2 \left[ \frac{q^2}{q^2 (Dq^2 + \delta)} \right]^2 = 9g^2 T \int_{\Lambda} \tilde{d}^4 q \left( \frac{1}{(Dq^2 + \delta)^2} \right) \\
\rightarrow & \text{ identical to Model A!}
\end{align*}
\]

\[\text{ii) The dynamics of Model A and Model B is different} \]

- modified dynamical exponents: the linearized dynamics is:

\[
\begin{align*}
\partial_t \varphi &= -\nabla^2 (-\nabla^2 D + \delta) \varphi + \ldots \\
i\omega \varphi &= q^2 (q^2 D + \delta) \varphi + \ldots
\end{align*}
\]

\[\Rightarrow \omega \propto q^4 \text{ at the critical point } \delta = 0 \text{ (NB: } \delta \text{ is still the tuning parameter of the phase transition [identical statics!], despite it takes the form of a diffusion term in Model B)} \]

\[\Rightarrow z = 4 \text{ for Model B (cf. } z = 2 \text{ in Model A!)}
\]

56
modified scaling: $z$ changes, the other exponents remain the same:

\[
\begin{align*}
\gamma: & \quad d + \bar{\chi} + \chi = 0 \quad \Rightarrow \quad z = 4 \text{ (cf. 2 in Model A)} \\
D: & \quad d + z - 4 + \bar{\chi} + \chi = 0 \quad \Rightarrow \quad \chi = \frac{2-d}{2} \\
\gamma T: & \quad d + z - 2 + 2\bar{\chi} = 0 \quad \Rightarrow \quad \bar{\chi} = -\frac{2+d}{2}
\end{align*}
\]

Loop corrections modify the value of $z$.

**summary:**

- identical statics \(\Rightarrow\) the upper critical dimension for Model B is identical to Model A despite the additional gradients
- but the dynamics is very different already on the "classical" level (no fluctuations)
- Model B constitutes a different dynamical universality class than Model A.
- the resulting hierarchical structure is called **dynamical fine structure**. It was established by Halperin and Hohenberg, enumerating Models A - J depending on various conservation laws.
- NB: a weak violation of the conservation law introduces terms without additional $\nabla^2$ in front. Those terms are more relevant \(\Rightarrow\) the RG flow quickly approaches the Model A fixed point in this case!

5.2 **Non-equilibrium phase transitions: Surface growth, KPZ equation**

- So far: near equilibrium dynamics/criticality. Equilibrium fixed point stable (static exponents coincide with purely static equilibrium problem).
- Now: strongly non-equilibrium situation (cf. exercises: we have seen that the KPZ nonlinearity is not of potential form). Described by strong coupling non-equilibrium fixed point different from equilibrium Wilson-Fisher FP.

5.2.1 Derivation from geometric considerations

- **point particles**: Brownian motion (diffusion), described by a local relaxing and noisy particle density $n$ in the continuum limit. Consider a random walk, $n_i$ a local density, particle number conserved: A particle lost at $i$ must reappear at $i \pm 1$, i.e.

$$\partial_t n_i(t) = (-n_i(t) + n_{i+1}(t)) + (-n_{i}(t) + n_{i-1}(t)) + \xi_i(t) = n_{i+1}(t) - 2n_i(t) + n_{i-1}(t) + \xi_i(t)$$

$$\Rightarrow \partial_t n(t, \vec{r}) = D \nabla^2 n(t, \vec{r}) + \xi(t, \vec{r})$$

- Q: what is the analog of Brownian motion of **surfaces** (strings, membranes)?

  - Turns out that random surface motion is qualitative distinct from point particles in the presence of **drive**

  - Simplest situation (Kardar, Parisi, Zhang 1986): Random particle deposition on tilted surface (growth) with rate $\lambda$

    assumption: growth proceeds along **normal** to the tilted plane: $ds = \lambda dt$

    - leads to dynamics of height variable

$$\partial_t h(\vec{r}, t) = D \nabla^2 h(\vec{r}, t) + \frac{\lambda}{2} (\nabla h(\vec{r}, t))^2 + \xi$$

(175)
derivation (one dimension):

\[ \frac{\partial h}{\partial r} = \tan \theta \quad 0 \leq \theta \leq \frac{\pi}{2} \]

\[ \Rightarrow \cos^2 \theta \left( \frac{dh}{dr} \right)^2 = 1 - \cos^2 \theta \Rightarrow \sqrt{1 + \left( \frac{dh}{dr} \right)^2} = \frac{1}{\cos \theta} \]

\[ \Rightarrow dh = \frac{ds}{\sqrt{1 + \left( \frac{dh}{dr} \right)^2}} \cong \lambda dt \]

\[ \Rightarrow \text{geometry induced contribution to the EoM:} \]

\[ \left. \frac{\partial h}{\partial t} \right|_{\text{geom.}} = \lambda \left( 1 - \frac{1}{2} \left( \frac{dh}{dr} \right)^2 \right) \]

\[ \Rightarrow \text{full EoM, } d \text{ dimensions} \]

\[ \frac{\partial h}{\partial t} = D \nabla^2 h + \lambda \left( \nabla h \right)^2 + \xi \]

simplifications/adjustments:

* transformation into comoving/rotating frame (comoving with surface height)
  \[ h(\vec{r}, t) \rightarrow h(\vec{r}, t) + \lambda t \text{ removes } \lambda \text{ from EoM} \]
* bring into canonical form via \( h \rightarrow -h \) (sign in front of \( \lambda \))

NB: The system is driven only for a finite tilt \( \theta \). Indeed, \( \frac{\partial h}{\partial r} = 0 \) for \( \theta = 0 \) \( \Rightarrow dh = ds \) and Brownian motion is recovered, and there is a balance of forces. Instead, the tilt induces a shear \( \frac{\partial h}{\partial r} \), resulting from the gravitational force \( \propto \vec{g} \)

5.2.2 Symmetries of the KPZ equation

- There is an interesting analogy to the physics of a phase which elucidates the symmetries:

  1. comoving frame transformation = rotating frame transformation = time-local gauge transformation

     - consider complex field, time-local gauge transformation

     \[ \varphi(\vec{r}, t) \rightarrow e^{i\lambda t} \varphi(\vec{r}, t) \]

     - in phase amplitude form:

     \[ \varphi(\vec{r}, t) = \rho(\vec{r}, t) e^{i\theta(\vec{r}, t)} : \quad \theta(\vec{r}, t) \rightarrow \theta(\vec{r}, t) + \lambda t \]
2. Galilean invariance

- Galilean transformation on complex field ($\vec{v}_0 = \frac{\vec{q}_0}{m}$):

$$\varphi(\vec{r},t) \rightarrow \varphi'(\vec{r}',t') = e^{i(E_0t - \vec{q}_0 \vec{r})} \varphi(\vec{r} - \frac{\vec{q}_0}{m} t, t) \quad (183)$$

$$E_0 = \frac{1}{2} m \vec{v}_0^2 = \frac{1}{2m} \vec{q}_0^2 \quad (184)$$

i.e. for the phase:

$$\theta(\vec{r},t) \rightarrow \theta'(\vec{r}',t') = \theta(\vec{r} - \frac{\vec{q}_0}{m} t, t) + E_0 t - \vec{q}_0 \vec{r} \quad (185)$$

- to make direct contact to KPZ we set $\frac{1}{m} = \lambda$. Then

$$\theta'(\vec{r}',t') = \theta(\vec{r} - \vec{q}_0 \lambda t, t) + \frac{\lambda}{2} \vec{q}_0^2 t - \vec{q}_0 \vec{r} \quad (186)$$

- this is a symmetry of the KPZ equation:

$$\partial_t \theta'(\vec{r}', t') = -\lambda \vec{q}_0 \nabla \theta(\vec{r}', t) + \partial_t \theta(\vec{r}', t)|_{\vec{r}'t'} + \frac{\lambda}{2} \vec{q}_0^2 \quad (187)$$

$$\overset{\dagger}{=} D \nabla^2 \theta(\vec{r}', t) + \frac{\lambda}{2} (\nabla \theta(\vec{r}', t) - \vec{q}_0)^2 + \xi(\vec{r}', t) \quad (188)$$

$$= D \nabla^2 \theta(\vec{r}', t) + \frac{\lambda}{2} (\nabla \theta(\vec{r}', t))^2 - \lambda \vec{q}_0 \nabla \theta(\vec{r}', t) + \frac{\lambda}{2} \vec{q}_0^2 + \xi(\vec{r}', t) \quad (189)$$

$$\Rightarrow \partial_t \theta(\vec{r}', t) = D \nabla^2 \theta(\vec{r}', t) + \frac{\lambda}{2} \nabla \theta(\vec{r}', t) + \xi(\vec{r}', t)$$

identical form of the original equation. In other words, the equation of motion depends on the following **invariant combination** alone,

$$\partial_t h - \frac{\lambda}{2} (\nabla h)^2 \quad (190)$$

- implication for the KPZ field theory defined with action

$$iS[h,p] = \int_{\vec{r},t} p \left[ \partial_t h - D \nabla^2 h - \frac{\lambda}{2} (\nabla h)^2 - \Delta p \right] \quad (191)$$

The action must depend on the invariant combination Eq. (190) alone. In particular, for the full effective action we may introduce multiplicative renormal-
\begin{align}
\int_{t,\vec{r}} p \, Z \, \partial_t h - \frac{\lambda}{2} \, Z'(\nabla h)^2
\end{align}

but implementing dependence on the invariant combination alone implies $Z = Z'$ (Ward identity).

3. Scale invariance = global gauge invariance

- in phase-only (Goldstone) action, scale invariance (the gapless nature of the action) is protected by the invariance under global phase shifts:

$$\varphi(\vec{r}, t) \to e^{i \alpha} \varphi(\vec{r}, t)$$

(193)

i.e. for the phase

$$\theta(\vec{r}, t) \to \theta(\vec{r}, t) + \alpha$$

(194)

- the deterministic part of the KPZ action involves only derivatives

$\Rightarrow$ Eq. (194) is a symmetry ($h(\vec{r}, t) \to h(\vec{r}, t) + \alpha$) $\Rightarrow$ no gap ($\sim \delta ph$) will be generated under RG. Important point: unlike critical phenomena, the gapless character results from the microscopic dynamics (diffusion term and the $\lambda$-term are gapless!) and not as a consequence of fine tuning! This is called "self-organized criticality" (similar: Kosterlitz-Thouless phase, non-Fermi liquids)

- Remark: to recover the invariances of the equation of motion, we need to transform $p(\vec{r}, t) \to p(\vec{r}', t)$ for the symmetries 1., 3., and $p(\vec{r}, t) \to p(\vec{r}', t')$ for 2.

- we can use RG concepts to evaluate the long distance physics of the KPZ equation. Canonical RG transformation:

$$\vec{r} \to b \vec{r}, \quad t \to b^z t; \quad h(\vec{r}, t) \to b^x h(b \vec{r}, b^z t), \quad p(\vec{r}, t) \to b^x p(b \vec{r}, b^z t)$$

(195)

- a typical observable is the variance of the height between two points in space and time. We expect scaling (in comoving frame)

$$H(\vec{r}, t) := \langle [h(\vec{r}, t) - h(0, 0)]^2 \rangle = r^{2x} f_{\text{KPZ}} \left( \frac{t}{\vec{r}^2} \right)$$

(196)
with $f_{\text{KPZ}}(y)$ a universal scaling function with asymptotes

\begin{align*}
  f_{\text{KPZ}}(y \to 0) &= \text{constant} \\  f_{\text{KPZ}}(y \to \infty) &\sim y^{2\chi} \Rightarrow H(\vec{r}, t) \sim t^{2\chi}
\end{align*}

- $\chi$ is called \textbf{roughness exponent}:
  
  $\chi > 0$: height variance grows with $\vec{r}, t$: "rough phase"
  
  $\chi < 0$: height variance shrinks with $\vec{r}, t$: "smooth phase"

\subsection*{5.2.3 Perturbative renormalization}

- first we use two structural properties of the KPZ equation to derive a nontrivial scaling relation (relation between exponents $z, \chi, \tilde{\chi}$)
  
  (i) absence of renormalization of time derivative term ($Z = 1$)
    
    - nonlinear vertex decomposition
      
      \begin{align*}
        (a) &- \frac{\lambda}{2} p_s (\nabla h_f)^2; & (b) &- \lambda p_s \nabla h_f \nabla h_s
      \end{align*}
      
    - $h_s$ appears as gradient only
      
      $\Rightarrow$ it is not possible to generate a term $\sim \int_{\vec{r}, t} p_s \partial_t h_s = - \int_{\vec{r}, t} \partial_t p_s \cdot h_s$
      
      $\Rightarrow$ $Z = 1, d_Z = 0$
      
      $\Rightarrow$ (with power counting) $d + z - z + 0 + \tilde{\chi} + \chi = d + \tilde{\chi} + \chi = 0$ exact scaling relation
  
  (ii) use above Ward identity $Z = Z'$
    
    $\Rightarrow$ $\lambda$ is not renormalized, i.e. $d_\lambda = 0$ as well
    
    $\Rightarrow$ $d + z - 2 + \tilde{\chi} + 2\chi = 0$ exact scaling relation
    
    Combining both equations, we get an exact relation for roughness and dynamical exponent,
    
    $\boxed{\chi + z = 2}$
    
- in contrast to these exact considerations, the RG flow of the couplings $D, \Delta$ has to be
computed perturbatively:

\[ \delta \Delta = -\frac{\lambda^2 \Delta^2}{4} \int_{\Lambda/b}^{\Lambda} d^4q \frac{q^4}{(Dq^2)^3} = \Delta \frac{\lambda^2 \Delta}{2D^3} \tilde{K}_d(1 - 1/b) \] (199)

\[ \delta D = \frac{\partial}{\partial k^2} \frac{\lambda^2 \Delta}{2D^2} \int_{\Lambda/b}^{\Lambda} d^4q \frac{1}{q^2} \left( \frac{(\vec{k}_z^2 - \vec{q}^2)}{q^2} - \frac{\vec{k}_z^2}{2} \right) = D \frac{\lambda^2 \Delta}{2D^3} \left( \frac{1}{d} - \frac{1}{2} \right) \tilde{K}_d(1 - 1/b) \] (200)

\[ \tilde{K}_d = \Lambda^{d-2} \pi^{-(d/2)} \Gamma(d/2) \] where \( \delta \Delta \) corresponds to the left diagram and \( \delta D \) to the right diagram in the figure below.

(c) (d)

- **RG equation (using the exact scaling relations, and \( b = e^{-l}, 1 - \frac{1}{b} \approx l \))**

\[ \partial_l \Delta = (z - d - 2\chi + \bar{\lambda}^2 \tilde{K}_d) \Delta, \quad \partial_l D = (z - 2 + \frac{2-d}{d} \bar{\lambda}^2 \tilde{K}_d) D \] (202)

with \( \bar{\lambda}^2 := \frac{\lambda^2 \Delta}{4D^3} \) (203)

- this can be combined into a **single** RG equation for \( \bar{\lambda}^2 \):

\[ \partial_t \bar{\lambda}^2 = \frac{\lambda^2}{4D^3} \partial_t \Delta - \frac{3\lambda^2 \Delta}{4D^4} \partial_t D = \left( \epsilon + \frac{1 - 2\epsilon}{1 - \epsilon/2} \bar{\lambda}^2 \tilde{K}_d \right) \bar{\lambda}^2 \] (204)

here, \( \epsilon = 2 - d \) : \( d_c = 2 \) is the critical dimension, see the integrals!

- Flow diagram
– $d \leq 2$ : flow to strong coupling. There is a fully attractive strong coupling fixed point. It has no relevant direction, reflecting the presence of an extended critical phase. It is not accessible perturbatively, but has been established using functional RG (L. Canet et al., Phys. Rev. Lett. 104, 150601 (2010)) and numerics. From these studies, $\chi > 0 \Rightarrow$ rough phase.

The absence of a perturbatively accessible fixed point is very different from the generic equilibrium problems.

– $d > 2$ : Again there is a fully attractive strong coupling fixed point. In addition, even in perturbative RG there is an unstable fixed point at $\bar{\lambda}_c = (|\epsilon| |\tilde{K}|^2)^{1/2}$

For $d > 2$, $\lambda < \bar{\lambda}_c$, the system goes to a smooth phase: $\lambda \to 0 \Rightarrow$ free theory fixed point $\Rightarrow$ $z = 2 \Rightarrow \chi = 1 - d/2 < 0 \Rightarrow$ smooth phase (equilibrium, free diffusion)

**Discussion:** some more properties of the KPZ equation

– exact determination of static properties in $d = 1$, claim: stationary distribution coincides with free theory (!)

$$P(h) = Z_{th}^{-1} \exp \left[ -\frac{D}{2\Delta} \int_r (\partial_r h)^2 \right]$$ (205)

– proof: cast problem into FPE:

$$\partial_t P(h, t) = \int_r \frac{\delta}{\delta h(r)} \left[ -(D\partial_r^2 h + \frac{\lambda}{2}(\partial_r h)^2)P(h, t) + \Delta \frac{\delta P(h, t)}{\delta h(r)} \right]$$ (206)

* for the free distribution, first and last term on rhs cancel
remains to be shown (surface terms assumed to vanish):

\[ 0 = -\frac{\lambda}{2} \int \frac{\delta}{\delta h} \left( (\partial_r h)^2 P(h, t) \right) \text{ansatz} = \frac{\lambda}{2} \int \left[ 2\partial_r^2 h - (\partial_r h)^2 \frac{D}{2\Delta} \partial_r^2 h \right] P(h, t) \quad (207) \]

\[ = \lambda \int \partial_r \partial_r h - \frac{D}{2\Delta} (\partial_r^2 h)(\partial_r h)^2] P(h, t) = \lambda \int \left\{ \partial_r [\partial_r h - \frac{D}{6\Delta} (\partial_r h)^3] \right\} P(h, t) \quad \text{r-independent} \]

\[ = 0 \]

- This does not work in higher dimension: the following is not a total derivative

\[ \sum_{\alpha\beta} (\partial_\alpha h \partial_\alpha h)(\partial_\beta \partial_\beta h) \]

- Implication

* **static** exponents Gaussian \( \Rightarrow \chi = 1 - d/2 \)
* **dynamic** exponent from exact scaling relation: \( \Rightarrow z = 3/2 \)
* should be compared to free theory \( z = 2 \) (diffusion). This anomalous diffusion \( z = \frac{3}{2} \) has been discussed and compared to \( z = 2 \) in the first lecture!

- Applications: the KPZ equation is a paradigm of non-equilibrium statistical mechanics, and more precisely, of driven (energy non-conserving) systems with interfaces. It governs, e.g.:

<table>
<thead>
<tr>
<th>interface</th>
<th>drive</th>
</tr>
</thead>
<tbody>
<tr>
<td>defect growth in liquid crystals</td>
<td>edge of defect electric field</td>
</tr>
<tr>
<td>growth of bacterial colonies</td>
<td>edge sugar</td>
</tr>
<tr>
<td>fire spreading</td>
<td>front oxygen</td>
</tr>
<tr>
<td>phase dynamics of exciton-polariton systems</td>
<td>phase pump laser</td>
</tr>
</tbody>
</table>

- We will come back to the last point in the context of quantum systems!
Part II

Quantum dynamical systems

6 Quantum master equation

6.1 Quantum master equation, Lindblad form

- setting: driven open quantum system

- total Hamiltonian: $H_T = H + H_I + H_B$

- with:
  
  $-H_S \sim \omega_0 \quad$ typical scale, arbitrary polynomial of system creation/annihilation operators
  
  $-H_B = \sum_i \sum_{\mu} \epsilon_{\mu,i} b_{\mu,i}^{\dagger} b_{\mu,i} \rightarrow \sum_i \int d\omega \nu(\omega) \epsilon(\omega)_i b_{\omega,i}^{\dagger} b_{\omega,i} \quad$ quadratic form, continuum of bath oscillators
  
  $-H_I = \sum_i \sum_{\mu} g_{\mu,i} \left( b_{\mu,i}^{\dagger} L_i + b_{\mu,i} L_i^{\dagger} \right) \rightarrow \sum_i \int d\omega \nu(\omega) g(\omega) \left( b_{\omega,i}^{\dagger} L_i + b_{\omega,i} L_i^{\dagger} \right)$

- $L_i$ can be an arbitrary polynomial in the system creation/annihilation operators, but the bath variables enter linearly in $H_I$ only. Thus the bath appears as a harmonic variable (linear plus quadratic terms)

- $b_{\mu}, b_{\mu}^{\dagger}$ are bath annihilation/creation operators. Each system operator $L_i$ couples to its own bath

- $\nu(\omega)$ is the bath density of states, a continuous function chosen on phenomenological grounds

- there will be three key approximations
i) Born-Markov: weak system-bath coupling, \( g(\omega)/\omega_0 \ll 1 \), and system and bath density matrices factorize

ii) the bath is much larger than the system, and the bath is in a stationary state unaffected by the state of the system

iii) rotating wave: fast scale in system, see below

- derivation of quantum master equation (QME)

  - starting point: Heisenberg-von Neumann equation for the total density matrix \( \rho_T \):
    \[
    \partial_t \rho_T(t) = -i[H, \rho_T(t)]
    \] (209)

  - (i) weak coupling \( \Rightarrow \) second order time-dependent perturbation theory in the system-bath coupling:
    * interaction picture:
      \[
      \tilde{O}(t) := e^{i(H+H_B)t} O e^{-i(H+H_B)t}; \quad \tilde{O}(0) = O
      \] (210)

    \( \Rightarrow \) equation of motion
    \[
    \partial_t \tilde{\rho}_T(t) = -i[\tilde{H}_I(t), \tilde{\rho}_T(t)],
    \] (211)

    formal solution:
    \[
    \tilde{\rho}_T = U(t) \tilde{\rho}_T(0) U^\dagger(t),
    \] (212)
    \[
    U(t) = \mathcal{T}[e^{-i \int_0^t dt' H_I(t')} ] = 1 - i \int_0^t dt' \tilde{H}_I(t') + (-i)^2 \int_0^t dt' \int_0^{t'} dt'' \tilde{H}_I(t') \tilde{H}_I(t'') + ...
    \]

    with time ordered exponential defined via its series expansion. Expanding the solution on both sides, to second order perturbation theory we get

    \[
    \tilde{\rho}_T(t) \approx \tilde{\rho}_T(0) - i \int_0^t dt' [\tilde{H}_I, \tilde{\rho}_T(0)] + (-i)^2 \int_0^t dt' \int_0^{t'} dt'' [\tilde{H}_I(t'), [\tilde{H}_I(t''), \tilde{\rho}_T(t'')]]
    \] (213)

    \( \Rightarrow \)
    \[
    \partial_t \tilde{\rho}_T(t) = -i[\tilde{H}_I(t), \tilde{\rho}_T(0)] - \int_0^t dt' [\tilde{H}_I(t), [\tilde{H}_I(t'), \tilde{\rho}_T(t')]]
    \] (214)
trace out bath: evolution equation for the system alone, \( \tilde{\rho}(t) := \text{tr}_B \tilde{\rho}_T(t) \).

* uses, without loss of generality, \( \text{tr}_B[\hat{H}_I(t), \rho_B(0)] = 0 \): interaction \( \hat{H}_I(t) \) has no diagonal elements in basis where \( H_B \) is diagonal (else redefine \( H_B, H \) to remove them)

\[ \Rightarrow \text{evolution equation for } \tilde{\rho}(t) = \text{tr}_B \tilde{\rho}_T(t) \]

\[ \partial_t \tilde{\rho}(t) = - \int_0^t dt' \text{tr}_B \left( [\hat{H}_I(t), [H_I(t'), \tilde{\rho}_T(t')]] \right) \quad (215) \]

* (i) and (ii): \( \tilde{\rho}_T(t) = \tilde{\rho}(t) \otimes \rho_B \) with \( \rho_B \) time independent (stationary state). The bath is not affected by the system, but the system is affected by the bath:

\[ \partial_t \tilde{\rho}(t) = - \int_0^t dt' \text{tr}_B \left( [\hat{H}_I(t), [\hat{H}_I(t'), \tilde{\rho}(t') \otimes \rho_B]] \right) \quad (216) \]

* (ii) Markov: bath has short time correlations:

\[ \text{tr}_B(b^\dagger(\tau)b(0)\rho_B) \equiv \langle b^\dagger(\tau)b(0) \rangle \sim e^{i\tau/\tau_0} \text{ with } \tau_0 = 1/\omega_0 \ll 1/g_\mu \quad (217) \]

\( g_\mu \) is the strength of the system-bath coupling, so sets the typical time scale \( 1/g_\mu \) of evolution of the system due to its coupling to the bath (Eq. (211)). A fast oscillation has the same effect as a fast damping: leads to decorrelation in time, i.e. acts as \( \delta(\tau) \). In the continuum, the above condition reads \( g(\omega)/\omega_0 \ll 1 \): this is again the Born-Markov approximation. Implications: (a) \( \tilde{\rho}(t') \approx \tilde{\rho}(t) \) does not change substantially on time scale \( \tau_0 \), (b) we can approximate \( \int_0^t dt' \approx \int_{-\infty}^t dt' \)

\[ \Rightarrow \partial_t \tilde{\rho}(t) = - \int_0^\infty d\tau \text{tr}_B \left( [H_I(t), [H_I(t-\tau), \tilde{\rho}(t) \otimes \rho_B]] \right) \]

(iii) Rotating wave approximation (RWA)

* for the sake of generality, we work in the basis where the system operators in \( H_I \) take the form of eigenoperators of the system Hamiltonian \( H \):

\[ H_I = \sum_m X_m^+ v_m \Gamma_m + X_m^- v_m^\dagger \Gamma_m^\dagger \quad (218) \]

here, we absorb the index \( i \) into the definition of \( X_m^\pm, \Gamma_m, \Gamma_m^\dagger \) are linear in the
bath creation/annihilation operators; \( v_m \sim \kappa_\mu \) encode the strength of system-bath coupling; \( X^\pm_m \) are eigenoperators of the system, defined by

\[
[H, X^\pm_m] = \pm \omega_m X^\pm_m
\]  

(219)

* inserting into Eq. (218), we get terms of the type

\[
- \sum_{m,n} X^+_m X^-_n e^{i(\omega_m t - \omega_n(t-\tau))} \hat{\rho}(t) \text{tr}_B \left( \Gamma_m(t) \Gamma_n^\dagger(t-\tau) \rho_B \right)
\]

bath correlator


time-translation invariant

RWA \\sim - \sum_{m} X^+_m X^-_m \bar{\rho}(t) \left[ \int_0^\infty d\tau e^{i\omega_m \tau} \langle \Gamma_m(\tau) \Gamma_n^\dagger(0) \rangle \right]

(220)

* the RWA assumption is \( \omega_m - \omega_n \gg v_m \). In this case, rapidly oscillating terms \( n \neq m \) can be discarded.

* collect all terms together, use the abbreviations \( (\gamma_m, \lambda_m \geq 0 \text{ for stability}) \)

\[
\gamma_m + i\delta_m = \int d\tau e^{i\omega_m \tau} \langle \Gamma_m(\tau) \Gamma_n^\dagger(0) \rangle
\]

\[
\gamma_m - i\delta_m = \int d\tau e^{-i\omega_m \tau} \langle \Gamma_m(0) \Gamma_n^\dagger(\tau) \rangle
\]

\[
\lambda_m + i\epsilon_m = \int d\tau e^{i\omega_m \tau} \langle \Gamma_m(\tau) \Gamma_n^\dagger(0) \rangle
\]

\[
\lambda_m - i\epsilon_m = \int d\tau e^{-i\omega_m \tau} \langle \Gamma_n^\dagger(0) \Gamma_m(\tau) \rangle
\]

QME in interaction picture for \( \bar{\rho}(t) := \text{tr}_B \hat{\rho}_I(t) = e^{-iHt} \rho e^{iHt} \) for time independent system Hamiltonian, and using again \( \hat{\rho}_I(t) = \hat{\rho}(t) \otimes \rho_B \)

\[
\partial_t \bar{\rho} = -i \sum_m \delta_m X^+_m X^-_m + \epsilon_m X^-_m X^+_m \bar{\rho} \]

(221)

\[
+ \sum_m \gamma_m \left( 2X^-_m \bar{\rho} X^+_m - X^+_m X^-_m \bar{\rho} - \bar{\rho} X^+_m X^-_m \right)
\]

\[
+ \sum_m \lambda_m \left( 2X^+_m \bar{\rho} X^-_m - X^-_m X^+_m \bar{\rho} - \bar{\rho} X^-_m X^+_m \right)
\]

(222)

* \( \delta_m, \epsilon_m \) are called Lamb shifts, coherent processes which renormalize the Hamil-
tonian $H \rightarrow H + \sum_m \delta_m X_m^+ X_m^- + \epsilon_m X_m^- X_m^+$. We move back to the Schrödinger picture and absorb them into the definition of $H$. In contrast, the evolution generated by the other terms in Eq. (222) is dissipative. The final QME in the Schrödinger picture is

$$\frac{\partial}{\partial t} \rho(t) = -i[H, \rho(t)] + \sum_m \gamma_m \left(2X_m^- \rho X_m^+ - \{X_m^+ X_m^-, \rho\}\right)$$
$$+ \sum_m \lambda_m \left(2X_m^+ \rho X_m^- - \{X_m^- X_m^+, \rho\}\right)$$

(223)

**Discussion:**

- the rhs is called **Liouvillian or Liouville operator**. It is a superoperator: includes left and right action onto an operator, the density matrix in this case.
- Mechanism how the system affects the bath: its correlators provide "mean fields" (encoded in Eq. (221)) for the evolution of the system
- the evolution is **trace preserving**: $\partial_t \text{tr} \rho = 0$ due to cyclic invariance. Mnemonic for the relative sign and factor $2$ in the dissipative part
- the evolution is **completely positive** for $\gamma_m, \lambda_m > 0 \forall$: non-negative eigenvalues of $\rho$ are mapped into non-negative eigenvalues by application of the infinitesimal time step, i.e. the Liouvillian, to the density matrix (Lindblad, Kossakowski 1976)
- dropping the RWA, the dissipative terms would take the more general form

$$\sum_{m,n} \left(\gamma_{m,n} \left(2X_m^- \rho X_n^+ - \{X_n^+ X_m^-, \rho\}\right)\right) + \lambda_{m,n} \left(2X_n^+ \rho X_m^- - \{X_m^- X_n^+, \rho\}\right)$$

(224)

with $\gamma_{m,n}, \lambda_{m,n}$ positive semidefinite matrices (positive definite eigenvalues) ensuring complete positivity. This is the **most general time local dynamics** that can act on a density matrix.
- The general and diagonal form are related by a unitary diagonalizing $\gamma_{m,n} = u_\gamma \gamma u_\gamma^\dagger, \gamma = \text{diag}(\gamma_m); \lambda_{m,n} = u_\lambda \lambda u_\lambda^\dagger, \lambda = \text{diag}(\lambda_m)$. It thus provides the same dynamics as a RWA QME with $\gamma, \lambda$.
- There is another symmetry (transformation that does not affect the dynamics of a
system)

\[ H \rightarrow H + e \mathbb{1} - i \sum_m (g_m^* X_m^- - g_m X_m^+) \]  
(225)

\[ X_m^+ \rightarrow X^+ + g_m^* \mathbb{1}, \; X_m^- \rightarrow X^- + g_m \mathbb{1}, \; e \in \mathbb{R}, \; g_m \in \mathbb{C} \]  
(226)

\[ (227) \]

\( e \mathbb{1} \) is the usual constant shift of energy admissible in a purely Hamiltonian system. The more general freedom tells us that energy is no longer well defined: It is dependent on the (arbitrary) choice of \( g_m \). This is rationalized as the underlying system is driven: energy is not well defined in the presence of an oscillatory, i.e. time dependent drive.

- the driven two level system

\[ - \omega_0 \text{ - level spacing} \]
\[ - \nu \text{ - laser frequency} \]
\[ - \Omega \text{ - Rabi frequency, laser intensity} \]

The reduction of an atom to a two-level system corresponds to the rotating wave approximation taken in step zero: \( |e\rangle \) is the excited state of the atom closest to the laser drive frequency, with \( \Delta = \omega_0 - \nu \ll \omega_0 \). It also assumes implicitly a strongly anharmonic atomic level structure, else multi-photon processes would occupy equally spaced levels.
system Hamiltonian (zero of energy at |g⟩)
\[ \tilde{H} = \omega_0 |\bar{e}\rangle \langle e| + e^{i\omega t} |\bar{e}\rangle \langle g| + e^{-i\omega t} |\bar{g}\rangle \langle e| \] (228)
we absorb the explicit time dependence into the states |\bar{e}\rangle to get a time-independent system Hamiltonian:
\[ \begin{pmatrix} |g\rangle \\ |e\rangle \end{pmatrix} = U \begin{pmatrix} |\bar{g}\rangle \\ |\bar{e}\rangle \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\omega t} \end{pmatrix}, \quad U^\dagger U = 1 \text{ unitary} \] (229)

– intermediate calculation: time-independent system Hamiltonian $H$:
\[ \partial_t \rho = -i \left[ \tilde{H}, \rho \right]; \quad H = U \tilde{H} U^\dagger, \quad \rho = U \tilde{\rho} U^\dagger; \] (230)
\[ \text{use } U U^\dagger = -U U^\dagger \text{ as } U^\dagger U = 1 \]
\[ \Rightarrow \partial_t \rho = -i \left[ \tilde{H} + iUU^\dagger, \rho \right] \equiv -i [H, \rho], \] (231)
\[ \Rightarrow H = (\omega_0 - \nu) |e\rangle \langle e| + \Omega (|e\rangle \langle g| + |g\rangle \langle e|) \]
\[ \equiv \Delta |e\rangle \langle e| + \Omega (|e\rangle \langle g| + |g\rangle \langle e|) \]
\[ = +\Delta \left( \frac{1}{2} (1 - \sigma_z) \right) + \Omega \sigma_x = -\frac{\Delta}{2} \sigma_z + \Omega \sigma_x \text{ up to irrelevant constant.} \] (232)

– $H_B = \sum_\mu \epsilon_\mu b_\mu^\dagger b_\mu$, $\rho_B = \prod_\mu \rho_\mu$, $\rho_\mu = e^{-\beta \epsilon_\mu b_\mu^\dagger b_\mu}$ thermodynamic equilibrium

– $\tilde{H}_I = |g\rangle \langle e| \sum_\mu g_\mu \tilde{b}_\mu + \text{h.c.} = \tilde{\sigma}^- \sum_\mu g_\mu \tilde{b}_\mu; \quad g_\mu \text{ real}$
\[ \Rightarrow H_I = \sigma^- \sum_\mu g_\mu b_\mu(t) + \text{h.c.} \quad b_\mu(t) = e^{i\omega t} \tilde{b}_\mu \]
\[ X_m = \sigma^-, \quad \Gamma_m = \sum_\mu h_\mu b_\mu(t), \quad \bar{\Gamma}_m = \sum_\mu g_\mu \bar{b}_\mu \quad \text{(only one single } m) \]

- one advantage of the above choice of basis is that the operators \( \sigma^- \) are the true eigenoperators of \( H \):

\[ [H, \sigma^-] = \Delta \sigma^- \quad (233) \]

\[ \Rightarrow \text{ the bath correlators are} \]

\[ \int_0^\infty d\tau e^{-i\Delta \tau} \langle \Gamma^\dagger(\tau)\Gamma(0) \rangle \]

\[ = \sum_{\mu,\nu} \int_0^\infty d\tau e^{-i\Delta \tau} g_\mu g_\nu \langle \bar{b}_\mu^\dagger(\tau)b_\nu(0) \rangle \]

\[ = \sum_\mu \int_0^\infty d\tau e^{-i\Delta \tau} g_\mu^2 e^{-i\omega \tau} \langle \bar{b}_\mu^\dagger(\tau)\bar{b}_\mu(0) \rangle \]

\[ = \sum_\mu g_\mu^2 \int_{-\infty}^\infty d\tau e^{-i(\Delta+\nu)\tau} \Theta(\tau) \langle \bar{b}_\mu^\dagger(\tau)\bar{b}_\mu(0) \rangle \]

\[ = \sum_\mu g_\mu^2 \int_{-\infty}^\infty d\tau e^{-i\omega_\tau} \Theta(\tau) \langle \bar{b}_\mu^\dagger(\tau)\bar{b}_\mu(0) \rangle \]

with \( \langle \bar{b}_\mu^\dagger(t)\bar{b}_\mu(0) \rangle = n(\epsilon_\mu)e^{i\epsilon_\mu t} \quad (234) \]

\[ \Rightarrow \int d\Omega \nu(\Omega)g_\mu^2(\Omega) \int d\tau e^{-i(\omega_0 - \Omega)\tau} \Theta(\tau)n(\Omega) \]

\[ = \int d\Omega \nu(\Omega)g_\mu^2(\Omega)n(\Omega) \left[ \pi\delta(\Omega - (\omega_0)) + i \frac{P}{\Omega - \omega_0} \right] \]

\[ = \pi\nu(\omega_0)g_\mu^2(\omega_0)n(\omega_0) + i \int d\Omega \nu(\Omega)g_\mu^2(\Omega)n(\Omega) \frac{P}{\Omega - \omega_0} \]

\[ \equiv \lambda + i\epsilon \quad (235) \]

- \( n \) is the Bose distribution of the bath (here: vacuum radiation field)
- the dissipative contribution \( \lambda \) involves the Bose distribution of the bath modes evaluated at the large frequency \( \omega_0 \)
- \( n(\omega_0) = (\exp(\omega_0/T) - 1)^{-1}; \quad \omega_0 \approx 10^{15} \text{ Hz}, \ T = 300 \text{ K} \approx 6 \cdot 10^{12} \text{ Hz} \quad (\hbar = c = 1) \quad \Rightarrow n(\omega_0) \approx 0 \]
only the spontaneous emission terms persist, no spontaneous up-conversion:

\[ \langle b^\dagger_\mu(\tau)b_\mu(0) \rangle \approx n(\omega_0) \approx 0 \quad \text{vs.} \quad \langle b_\mu(\tau)b^\dagger_\mu(0) \rangle \approx n(\omega_0) + 1 \approx 1 \quad (236) \]

– for baths at equilibrium,

\[ \frac{\gamma}{\lambda} = \frac{n(\omega_0) + 1}{n(\omega_0)} > 1 \quad \text{"more losses than pumping"} \quad (237) \]

– for more general baths, or additional drives, there is no bound on \( \gamma/\lambda \)

• Summary: two level systems as a paradigm of driven open quantum systems

  – drive is essential to generate dynamics in the two level system
  – key implications: non-equilibrium character

    (i) no guarantee for detailed balance: the rates of transition between the levels do not exclusively depend on the energy differences \( \omega_0 = \omega_1 - \omega_0 \)

    (ii) no guarantee for the obedience of the second law of thermodynamics of the system density matrix

  – agenda for the following: we will study both these implications in a many-body context

6.2 Order by dissipation: purification in quantum many-body systems

• this catches up on point (ii) above

• the question:

  – usually: dissipation increases entropy, \( \partial_t S = \partial_t \text{tr} \rho \log \rho > 0 \) (classical example: friction, rubbing hands). This is the second law of thermodynamics

  – Q1: Is \( \partial_t S < 0 \) possible?

    A1: Yes, this is what happens in a fridge
Q2: Is it possible to create subtle quantum mechanical order by means of suitable chosen effective dissipative dynamics?

A2: Yes, if coherent drive is suitably combined with dissipation. More precisely, the following forms of quantum mechanical order by dissipation were demonstrated

- long range phase coherence (theory + experiment)
- entanglement (theory + experiment)
- topological order (theory)

- The setting: Lindblad quantum master equation (bath effectively at $T = 0$)

\[
\partial_t \rho = -i[H, \rho] + \kappa \sum_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \left\{ L_i^\dagger L_i, \rho \right\} \right) \equiv \mathcal{L}[\rho] \tag{238}
\]

- Key concept: dark states $|D\rangle$ (associated density matrix $\rho_D = |D\rangle \langle D|)$, defined by property

\[
H |D\rangle = E |D\rangle \quad \& \quad L_i |D\rangle = 0 \quad \forall i \tag{239}
\]

$\Rightarrow$ dark states are zero modes of the Liouvillian, $\mathcal{L}[\rho_D] = 0$, or dynamical fixed points

- Interesting situation: unique dark state

(a) dark subspace one-dimensional

(b) there are no other stationary solutions (this could arise, for example, by cancellation of the different terms in $\mathcal{L}$ without being zero individually)
key implication of a unique dark state solution:

- directed motion in Hilbert space, starting from arbitrary $\rho$, $\rho(t) = \sum_n p_n |\psi_n\rangle \langle \psi_n| \overset{t \to \infty}{\to} |D\rangle \langle D| \Rightarrow$ thinning out the density matrix to a pure quantum state even in an exponentially large Hilbert space of a many-body system

perspectives:

- targeted cooling of many-body systems of ultracold atoms (could be crucial for fermions)
- interesting non-equilibrium quantum physics

we will now study several examples in increasing complexity, from few to many degrees of freedom:

1. Damped cavity: single damped and oscillating bosonic mode

$$H = \omega_0 a^\dagger a, \quad L_i = a, \quad |D\rangle = |0\rangle \quad \text{empty cavity, trivial dark state} \quad (240)$$

2. Three-level system
\[ H = \Delta |e\rangle \langle e| + \Omega (|e\rangle \langle g_1| + e^{i\alpha} |e\rangle \langle g_2| + \text{h.c.}) \]

\[
= \begin{pmatrix} |g_1\rangle & |g_2\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} 0 & 0 & e^{i\alpha}\Omega \\ 0 & 0 & \Omega \\ e^{-i\alpha}\Omega & \Omega & \Delta \end{pmatrix} \begin{pmatrix} \langle g_1| \\ \langle g_2| \\ \langle e| \end{pmatrix}
\] (241)

\[ L_1 = |g_1\rangle \langle e|, \quad L_2 = |g_2\rangle \langle e|, \quad \text{rates } \gamma_1 = \gamma_2 = \gamma \]

\[ \Rightarrow |D\rangle = \frac{1}{\sqrt{2}} (|g_1\rangle - e^{-i\alpha} |g_2\rangle) \] (242)

in figure: \( \alpha = \pi \), \(|D\rangle = \frac{1}{\sqrt{2}} (|g_1\rangle + |g_2\rangle) \) for antisymmetric drive. The dark state is the symmetric superposition of the bare ground states: criteria (a) and (b) are fulfilled \( \Rightarrow \) quantum coherence (over the ground state levels) is generated by the combination of drive (\( \Omega \)) and dissipation (\( \gamma \), spontaneous emission rate)

3. From few to many levels: Dissipatively induced Bose condensation

- idea: replace internal degrees of freedom (atomic energy levels) by external (motional) degrees of freedom (sites in an optical lattice)

- 1 boson, 2 (lower) sites: (boson creation and annihilation operators \( a_i^\dagger, a_i \))

\[ \rightarrow \text{effective three-level system} \]

\[ \rightarrow 1 \text{ boson on this unit cell: same logic as above,} \]

\[ |D\rangle = \frac{1}{\sqrt{2}} (|g_i\rangle + |g_{i+1}\rangle) = \frac{1}{\sqrt{2}} \left( a_i^\dagger + a_{i+1}^\dagger \right) |0\rangle \]

\[ \] (244)

- advantage of external degrees of freedom: can be copied in translationally invariant way (optical lattice)

- \( N \) bosons, \( M \) sites, \( N, M \rightarrow \infty \): exponentially large Hilbert space
but we are still **locally** locking the phase into symmetric superposition

\[ |D \rangle = \frac{1}{\sqrt{N!}} \left( \frac{1}{\sqrt{M}} \sum_{i=1}^{M} a_i^\dagger \right)^N |0 \rangle = \frac{1}{\sqrt{N!}} a^{1N}_{q=0} |0 \rangle = |\text{BEC, } N \rangle \] (245)

\( a^{1}_{q=0} \) creates a particle in a zero momentum state (symmetric superposition over whole lattice): For \( N, M \to \infty \), this is a BEC state with a fixed particle number!

- we make these qualitative considerations more rigorous:
  * we introduce Lindblad operators in second quantization acting on pairs of sites (one dimension for simplicity)

\[ L_i = C_i^\dagger A_i, \quad C_i = a_i^\dagger + a_{i+1}^\dagger, \quad A_i = a_i - a_{i+1}. \] (246)

They transfer any antisymmetric component of a wave function on a pair of sites into the symmetric superposition on that pair. They thus do not change the number of particles, \( [L_i, \hat{N}] = 0 \), with the total particle number operator \( \hat{N} = \sum_i a_i^\dagger a_i \).

* To show that \( |\text{BEC, } N \rangle \) is a dark state, write \( |D \rangle = \frac{1}{\sqrt{N!}} (G^\dagger)^N |0 \rangle \) with generator \( G^\dagger = \left( \frac{1}{\sqrt{M}} \sum_{i=1}^{M} a_i^\dagger \right) \). We note the equivalence

\[ L_i |D \rangle = 0 \text{ for all } i \iff [L_i, G^\dagger] = 0 \text{ and } L_i \text{ normal ordered for all } i \] (247)

This holds for any state written in terms of a generator, and is easily verified in the present case.

* with more effort, it can also be shown that the dark state is unique for a fixed particle number \( N \) (criteria (a) and (b) above are fulfilled, cf. S. Diehl et al., Nature Physics 4, 878 (2008) and B. Kraus et al., PRA 78, 042307 (2008)).

* Thus, we are guaranteed that the density matrix is thinned out to a pure state, \( \rho(t \to \infty) \to |\text{BEC, } N \rangle \langle \text{BEC, } N| \).

- late time dynamics: Fixed number vs. fixed phase. We want to assess the late time evolution, where the density matrix is already close to the dark state.
* expand boson operators $a_i = a_0 + \delta a_i \Rightarrow L_i \approx 2a_0^*(\delta a_i - \delta a_{i+1}) = 2a_0^* \nabla_i \delta a_i$ in linearized approximation. Here we have assumed the equivalence of an ensemble with fixed number $N$ and one with a fixed phase $\theta$ (see below), as we allow the Bose operator to acquire a finite expectation value. This holds in the thermodynamic limit, where number fluctuations scale to zero ($(\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2) / \langle \hat{N} \rangle^2 \sim N^{-1}$ for the average taken with respect to the fixed phase dark state wavefunction).

* linearized dynamics

$$\partial_t \rho = 2\gamma a_0^* a_0 \sum_i \left( \nabla_i \delta a_i \rho \nabla_i \delta a_i^\dagger - \frac{1}{2} \{ \nabla_i \delta a_i^\dagger \nabla_i \delta a_i, \rho \} \right)$$

$$= 4\gamma a_0^* a_0 \sum_q (1 - \cos q) \left( \delta a_q \rho \delta a_q^\dagger - \frac{1}{2} \{ \delta a_q^\dagger \delta a_q, \rho \} \right)$$

(248)

* effective damping rate: $\gamma_q = 4\gamma a_0^* a_0 (1 - \cos q) \approx 2\gamma a_0^* a_0 q^2$ for $q \rightarrow 0$

$$\Rightarrow$$ dark state at $q = 0$ (no damping), slow damping in vicinity

* fixed phase dark state: coherent state as Fourier transform of fixed number state,

$$|\text{BEC, } \theta \rangle \propto e^{i\theta G^\dagger} |0\rangle = \sum_N e^{i\theta N} \frac{1}{\sqrt{N!}} G^{\dagger N} |0\rangle = \sum_N e^{i\theta N} |\text{BEC, } N\rangle$$

(249)

* indeed, this is a dark state of linearized evolution (recall $\gamma_{q=0} = 0$):

$$a_q |\text{BEC, } \theta \rangle = \delta_{q,0} e^{i\theta} |\text{BEC, } \theta \rangle$$

(250)

* NB: we could have cooled into any mode $q_*$ (no energy to be minimized) via

$$a_i - a_{i+1} \rightarrow (a_i - e^{iq_* x_i} a_{i+1})$$

• physical implementation
- **geometric setup**: optical superlattice, lower sites: target system, upper sites: auxiliary sites, integrated out

- **coherent drive**: drive laser with double wavelength as optical lattice \((\Omega_i = \Omega_{i+1}e^{i\pi} = -\Omega_i, \text{ phase shift})\). Site \(i, i+1\):

  \[
  H_{\text{drive}} = \Omega b_i^\dagger a_i + (-\Omega) b_i^\dagger a_{i+1} + \text{h.c.} = \Omega b_i^\dagger (a_i - a_{i+1}) + \text{h.c.} \quad (251)
  \]

- **dissipative process**: bath immersion into BEC reservoir: spontaneous emission

  \[
  H_I = \sum_k (r_k + r_k^\dagger) b_i (a_i^\dagger + a_{i+1}^\dagger) + \text{h.c.}, \quad H_B = \sum_k \omega_k r_k^\dagger r_k \quad (252)
  \]

  phonon bath \((\omega_k \sim 10^3\text{Hz}, \text{ cf. photon bath: } \omega_k \sim 10^{15}\text{Hz})\); for \(\lambda_{\text{phonon}} \gtrsim \lambda\), the spontaneous emission occurs indistinguishably over two sites

- scale hierarchy: \(\omega_0 \gg \Delta \gg (\text{other scales})\). Implications:
  
  * \(n_{\text{bath}} \left(\frac{\omega_0}{T_{\text{bath}}}\right) \approx 0\) for \(\omega_0 \gg T_{\text{bath}}\) \(\Rightarrow\) we can cool to pure states although the bath is at finite temperature!
  
  * \(\Delta \gg (\text{other scales})\) allows us to integrate out the \(b_i\) levels to obtain effective dynamics of target system operators alone:

  \[
  L_i = C_i^\dagger A_i = (a_i^\dagger + a_{i+1}^\dagger) (a_i - a_{i+1}) \quad (253)
  \]
4. Fermions: Dissipatively induced pairing mechanism

- motivation: cooling of fermions in optical lattices very hard, major roadblock for eg. quantum simulation of the Fermi-Hubbard model
- goal: reach low entropy ordered states of fermions (including topological states)
- simplest example: reinterpret bosons \([a_i, a_j^\dagger] = \delta_{ij}\) as fermions \(\{a_i, a_j^\dagger\} = \delta_{ij}\)
- intriguingly, there is an exact unique dark state wave function, which is a BCS pairing state with fixed pair number. Pairing is imposed due to single particle dissipative operations acting on the **density matrix without any attractive forces**!
- idea:
  - number-phase duality:
    \[
    L_i = C_i^\dagger A_i \quad \Leftrightarrow \quad \ell_i = C_i^\dagger + e^{-i\theta} A_i
    \]
    \[
    |\text{BCS}, N\rangle = \frac{1}{\sqrt{N}} G^N |0\rangle \quad |\text{BCS}, \theta\rangle = \sum_N e^{i\theta N} |\text{BCS}, N\rangle
    \]
    \[
    L_i |\text{BCS}, N\rangle = 0 \forall i \quad \ell_i |\text{BCS}, \theta\rangle = 0 \forall i
    \]
  - here, \(G^\dagger\) creates a fermionic Cooper pair \(G^\dagger = \sum_q \varphi_q a_q^\dagger a_{-q}\) (value of wave function \(\varphi_q\) : see next)
  - we show: "\(\Leftarrow\)" of the duality:
    * recall the usual BCS ground state condition: Any translation invariant quadratic Hamiltonian of fermions can be diagonalized by a unitary canonical transformation to the Bogoliubov basis (we stick to one dimension for simplicity), \(H = \sum_q \ell_q^\dagger \ell_q\), with the action of the Bogoliubov eigenoperators on the ground state with a fixed phase \(|\text{BCS}, \theta\rangle\)
    \[
    \ell_q |\text{BCS}, \theta\rangle = 0
    \]
    * Here, these operators are the (Fourier transforms of) \(\ell_i = (a_i^\dagger + a_{i+1}^\dagger) + (a_i - a_{i+1})\) (we set \(\theta = 0\) for simplicity)
    \[
    \Rightarrow \ell_q = u_q a_q + v_q a_{-q}^\dagger, \quad u_q = 2 \cos q, \quad v_q = 2i \sin q
    \]
    * \(|\text{BCS}, \theta\rangle = e^{i\theta G^\dagger} |0\rangle = \prod_q e^{i\theta \varphi_q a_q^\dagger a_{-q}} |0\rangle = \prod_q \left(1 + e^{i\theta \varphi_q a_q^\dagger a_{-q}}\right) |0\rangle\)
    * it is easy to verify \(\ell_q |\text{BCS}, \theta\rangle = 0 \forall q \) for \(\varphi_q = \frac{u_q}{v_q}\) in this representation
this implies \( \ell_i |\text{BCS}, \theta \rangle = 0 \forall i \Rightarrow C_i^\dagger |\text{BCS}, \theta \rangle = -A_i |\text{BCS}, \theta \rangle \Rightarrow L_i |\text{BCS}, \theta \rangle = -C_i^2 |\text{BCS}, \theta \rangle = 0 \forall i \) (Pauli principle)

* number conservation \([L_i, \hat{N}] = 0 \Rightarrow \text{the dark state condition holds for each particle number sector separately} \Rightarrow L_i |\text{BCS}, N \rangle = 0 \forall i \): these states are exact dark states of \( L_i \)!

- late time behavior:

* using ensemble equivalence as above, it can be shown that the operators \( L_i \) linearize as

\[
L_i = C_i^\dagger A_i \overset{t \to \infty}{\longrightarrow} \ell_i = C_i^\dagger + A_i
\]  

(256)

* effective late time dynamics in Fourier space:

\[
\partial_t \rho = \sum_q \gamma_q (\ell_q \rho \ell_q^\dagger - \frac{1}{2} \{\ell_q^\dagger \ell_q, \rho\})
\]  

(257)

* crucially, unlike for bosons but in analogy to BCS superconductors, a scale \( \gamma_* \) is generated in the long time limit (dissipative gap)

\[
\text{this means that the approach to the stationary state is exponentially fast (unlike bosons: polynomial behavior)}
\]

7 Keldysh field theory

7.1 Keldysh functional integral for open quantum systems

Basic idea in three steps:

1) Schrödiger equation evolves state vector

\[
\text{differential form} \quad i\partial_t |\psi\rangle (t) = H |\psi\rangle (t) \quad \Rightarrow \quad |\psi\rangle (t) = U(t, t_0) |\psi\rangle (t_0)
\]

\[
U(t, t_0) = e^{-iH(t-t_0)}
\]
2) Heisenberg-von Neumann equation evolves state (density) matrix

\[ \partial_t \rho(t) = -i[H, \rho(t)] \Rightarrow \rho(t) = U(t, t_0) \rho(t_0) U(t, t_0)^\dagger \]

NB: identical for pure states: \( \rho = |\psi\rangle \langle \psi| \)

3) the same is true for the quantum master equation (QME)

\[ \partial_t \rho(t) = -i[H, \rho(t)] + \sum_i \kappa_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \{ L_i^\dagger L_i, \rho \} \right) \]

\[ \equiv \mathcal{L}(\rho) \quad \text{Liouvillian superoperator} \tag{258} \]

\[ \Rightarrow \rho(t) = e^{(t-t_0)\mathcal{L}} \rho(t_0) \tag{259} \]

- assumed \( T_{\text{bath}} = 0 \) for simplicity
- \( \mathcal{L} \) acts on both sides of \( \rho \)
- the most general (time-local) quantum evolution transforms \( \rho \) as a matrix

Sketch of the functional integral derivation along the three steps:

1) functional integral idea:

- Trotter decomposition of the evolution operator and insertion of coherent states (cf. discussion of MSR vs. FPE), graphically

Formally

- preparation:
  - bring \( H \) into **normal order**, eg. single boson degree of freedom:

\[ H = \omega_0 a^\dagger a + \lambda \hat{n}(\hat{n} - 1) = \omega_0 a^\dagger a + \lambda (a^\dagger)^2 a^2 \quad \text{normally ordered} \tag{260} \]

  - coherent state: \( a |\phi\rangle = \phi |\phi\rangle \quad \text{(NB: } a^\dagger |\phi\rangle \neq \phi^* |\phi\rangle!) \)

  - overlap: \( \langle \phi'|\phi\rangle = e^{\phi^*\phi} \)

  - \( \mathbb{1} = \int \frac{d\phi}{2\pi} e^{-\phi^* \phi} |\phi\rangle \langle \phi| \quad \text{completeness relation} \)

- one single time step
evaluate the matrix element:

\[
e^{-\phi_n^* \phi_n} \langle \phi_{n+1} | e^{-i \delta_t H[a^\dagger, a]} | \phi_n \rangle \approx e^{-\phi_n^* \phi_n} (1 - i \delta_t H[a^\dagger, a] | \phi_n \rangle
\]

\[
H \text{ normally ordered} = e^{-\phi_n^* \phi_n} e^{\phi_{n+1}^* \phi_n} (1 - i \delta_t H[\phi_{n+1}^*, \phi_n])
\]

\[
\approx e^{\phi_{n+1}^* \phi_n - \phi_n^* \phi_n - i \delta_t H[\phi_{n+1}^*, \phi_n]}
\]

\[
e^{\delta_t \frac{\partial}{\partial t} \phi_n^* (t) - \phi_n (t) - H[\phi^* (t), \phi (t)]}
\]

\[
\text{continuum limit } \delta_t \to 0
\]

\[
- \text{many time steps (finite interval)}
\]

\[
\int \prod_t \frac{d\phi^*(t) d\phi(t)}{\pi} \left( i \int_{t_i}^{t_f} dt \left[ -i \partial_t \phi^* (t) \cdot \phi (t) - H[\phi^* (t), \phi (t)] \right] \right)
\]

\[
\text{NB:}
\]

* operator \( H[a^\dagger, a] \to H[\phi^* (t), \phi (t)] \) complex functional

* time evolution from overlap of neighboring states

2) generalize to matrix evolution and define "partition function"

- "trotterize" and insert coherent states as above

- but have to act on both sides of \( \rho \):
NB: this needs two sets of degrees of freedoms for the matrix evolution $\phi_{n,+}, \phi_{n,-}$

- finally, we are interested in a "partition function":

$$Z = \text{tr}\rho(t) = \text{tr}U\rho(t_0)U^\dagger = \text{tr}\rho(t_0) = 1 \quad (263)$$

- the trace operation contracts the evolution times:

- information on all stages: send $t_0 \rightarrow -\infty$, $t_f \rightarrow +\infty$

3) application to Liouville superoperator (= functional integral quantization of QME)

$$\partial_t \rho = \mathcal{L}[\rho] = \left(-iH - \frac{1}{2} \sum_i \kappa_i L_i^\dagger L_i \right) \rho + \rho \left(+iH - \frac{1}{2} \sum_i \kappa_i L_i^\dagger L_i \right) + \sum_i \kappa_i L_i \rho L_i^\dagger \quad (264)$$

- normal order on each side of $\rho$

  - in particular, eg. $L_i = \hat{n}_i$; $L_i^\dagger L_i = \hat{n}_i^2 = (a_i^\dagger)^2 a_i^2 + a_i^\dagger a_i$

- formal solution

$$\rho(t) = e^{(t-t_0)\mathcal{L}[\rho]} \overset{\text{def}}{=} \lim_{N \rightarrow \infty} (1 + \delta_t \mathcal{L})^N \rho(t_0); \quad \delta_t = \frac{t - t_0}{N} \rightarrow 0 \quad (265)$$

- in each time step, apply superoperator according to Eq. (264)

- Trotterize, insert coherent states, partition function

$$Z = \text{tr}\rho(t_f) = \text{tr} \lim_{N \rightarrow \infty} (1 + \delta_t \mathcal{L})^N \rho(t_0), \quad t_0 \rightarrow -\infty, t_f \rightarrow +\infty \quad (266)$$
result (lattice system)

\[ Z = \text{tr} \rho(t) = \int \mathcal{D}(\Phi_+, \Phi_-) e^{iS[\Phi_+, \Phi_-]} = 1 \]  

with \( \Phi_{\pm} = \left( \begin{array}{c} \phi_{\pm i}(t) \\ \phi^*_{\pm i}(t) \end{array} \right) \), \( \mathcal{D}(\Phi_+, \Phi_-) = \prod_{\sigma = \pm} \prod_{t} \prod_{i=1}^{M} \int \frac{d\phi^*_{\sigma i}(t) d\phi_{\sigma i}(t)}{\pi} \)  

\[ S = \int_{t_{0} \to -\infty}^{t_{\to +\infty}} dt \left[ \sum_{i} \phi^*_{+ i}(t) i \partial_t \phi_{+ i} - \phi^*_{- i} i \partial_t \phi_{- i}(t) - i \mathcal{L}[\Phi_+, \Phi_-] \right] \]  

\[ \mathcal{L}[\Phi_+, \Phi_-] = -i(H_+ - H_-) + \sum_{\sigma} \kappa_{\sigma} \left( L_{+ i} L_{- i}^\dagger - \frac{1}{2} L_{+ i} L_{i+} - \frac{1}{2} L_{i-}^\dagger L_{i-} \right) \]  

\[ H_{\pm} = H[\Phi_{\pm}] \] etc.  

\[ \Rightarrow \text{simple translation table}^{2} \text{ QME} \to \text{Keldysh functional integral, which works as follows:} \]

- normal order operators
- then:
  - operator left of \( \rho \to + \) contour
  - operator right of \( \rho \to - \) contour

how to extract physical information? \( (Z = 1 !) \)

- introduce source terms \( J_{\sigma} = \left( j_{\sigma i}(t), \ j^*_{\sigma i}(t) \right)^T \):

\[ Z[J_+, J_-] = \int \mathcal{D}(\Phi_+, \Phi_-) e^{iS[\Phi_+, \Phi_-] + \sum_{\sigma = \pm, i} \int \sigma J^*_{\sigma i}(t) \Phi_{\sigma i}(t)} = \left\langle e^{i \sum_{\sigma, i} \int \sigma J^*_{\sigma i}(t) \Phi_{\sigma i}(t)} \right\rangle \]  

\[ Z[0, 0] = 1 \]  

Source terms may be thought of as terms adding to the Hamiltonian: \( H \to H + \sum_{i} j_{i} \phi^*_i + j^*_i \phi_i \) in the second quantized representation. This explains their Keldysh contour structure.

This generates so-called contour ordered correlation functions. A more efficient

\[^{2}\text{Disclaimer: subtleties may arise for the contour coupling term. This can be cured by starting from } H_{\text{tot}} \text{ as in the derivation of the quantum master equation and tracking time regularization. See L. Sieberer et al., PRB 89, 134310 (2014).}\]
representation is obtained via the change of basis referred to as “Keldysh rotation”
\[
\begin{pmatrix}
\phi_c \\
\phi_q
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\phi_+ + \phi_- \\
\phi_+ - \phi_-
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
\phi_+ \\
\phi_-
\end{pmatrix}
\]  
"classical" field:

"quantum" field: relative coordinate

(274)

- the notation “quantum” and “classical” derives from the following consideration:
  - the + and - contours represent the same physics, just in a contour conjugated way. Therefore, for observable expectation values we must have, in the absence of sources, \( \langle \phi_+(t) \rangle = \langle \phi_-(t) \rangle \), which may be nonzero. Thus \( \langle \phi_c(t) \rangle \sim \langle \phi_+(t) \rangle + \langle \phi_-(t) \rangle \) may be nonzero (“classical field”), while \( \langle \phi_q(t) \rangle \sim \langle \phi_+(t) \rangle - \langle \phi_-(t) \rangle = 0 \) always (“quantum field”)

- NB: \( \phi_c(\phi_q) \) is even (odd) under contour exchange + ↔ −

- Terminology: \( c,q \) basis, Keldysh basis, RAK basis

- This representation is more efficient for two reasons
  - (i) it removes a redundancy in the Green’s functions
  - (ii) it generates the two basic types of physical observables: correlation and response functions

- we elaborate on these points in the next section, preparing the discussion with two important formulas:
  - partition function in the Keldysh basis (omit space index for simplicity)
    \[
    Z[J_c, J_q] = \left\langle e^{i \int (J_c \phi_c + J_q \phi_q)} \right\rangle
    \]  
    (275)

  - (disconnected) Green’s function in the Keldysh basis
    \[
    \begin{pmatrix}
    \langle \phi_c(t) \phi_c^*(t') \rangle \\
    \langle \phi_q(t) \phi_q^*(t') \rangle
    \end{pmatrix} = - \begin{pmatrix}
    \delta^2 Z_{J_c(t)J_c(t')} \\
    \delta^2 Z_{J_q(t)J_q(t')}
    \end{pmatrix} \begin{pmatrix}
    \delta^2 Z_{J_c(t)J_q(t')} \\
    \delta^2 Z_{J_q(t)J_q(t')}
    \end{pmatrix} \bigg|_{j_c=j_q=0}
    \]
    \[
    =: i \begin{pmatrix}
    G^K(t-t') & G^R(t-t') \\
    G^A(t-t') & 0
    \end{pmatrix}
    \]  
    (276)

7.2 General properties of Keldysh field theories

- goals: understand the properties
- preservation of probability (also sometimes referred to as causality structure)
- correlation vs. response functions
- basic analyticity properties of single particle Green’s functions

• We will make these properties manifest by using the simple example of a damped quantum cavity, but point out that they are general (present also in interacting many body problems) alongside

• model: pumped ($\gamma_p$), lossy ($\gamma_l$) quantum cavity (frequency $\omega_0$)

$$\partial_t \rho = -i[\omega_0 a^\dagger a, \rho] + \gamma_p (2a^\dagger a - \{aa^\dagger, \rho\}) + \gamma_l (2\rho a^\dagger - \{a^\dagger a, \rho\})$$  \hspace{1cm} (277)

• action

$$S = \int dt \left( a^*_+(t), a^*_+(t) \right) \begin{pmatrix} i\partial_t - \omega_0 + i(\gamma_p + \gamma_l) & -2i\gamma_p \\ -2i\gamma_l & -i\partial_t - \omega_0 - i(\gamma_p + \gamma_l) \end{pmatrix} \begin{pmatrix} a_+(t) \\ a_-(t) \end{pmatrix}$$

$$= \int dt \left( a^*_e(t), a^*_q(t) \right) \begin{pmatrix} 0 & i\partial_t - \omega_0 - i(\gamma_l - \gamma_p) \\ i\partial_t - \omega_0 + i(\gamma_l - \gamma_p) & 2i(\gamma_l + \gamma_p) \end{pmatrix} \begin{pmatrix} a_e(t) \\ a_q(t) \end{pmatrix}$$

$$= \int \frac{d\omega}{2\pi} \left( a^*_e(\omega), a^*_q(\omega) \right) \begin{pmatrix} 0 & P^A(\omega) \\ P^R(\omega) & P^K(\omega) \end{pmatrix} \begin{pmatrix} a_e(\omega) \\ a_q(\omega) \end{pmatrix}$$  \hspace{1cm} (280)

with $P^R(\omega) = P^A(\omega)^* = \omega - \omega_0 + i\kappa$, $P^K = 2i\bar{\kappa}$, $\kappa = \gamma_l - \gamma_p$, $\bar{\kappa} = \gamma_l + \gamma_p$

• This matrix is the inverse single-particle Green’s function. More precisely, in the frequency domain,

$$\tilde{G}^{-1}(\omega, \omega') = G^{-1}(\omega)\delta(\omega - \omega'), \quad G^{-1}(\omega) = \begin{pmatrix} 0 & P^A(\omega) \\ P^R(\omega) & P^K(\omega) \end{pmatrix}$$  \hspace{1cm} (281)

• The single particle Green’s function can be computed exactly for the free theory:

$$Z = \exp \left[ i \int_{\omega,\omega'} \left( j^*_q(\omega), j^*_e(\omega) \right) G(\omega, \omega') \begin{pmatrix} j_q(\omega') \\ j_e(\omega') \end{pmatrix} \right]$$  \hspace{1cm} (282)
with

\[
\tilde{G}(\omega, \omega') = \begin{pmatrix}
G^K(\omega) & G^R(\omega) \\
G^A(\omega) & 0
\end{pmatrix} \delta(\omega - \omega'), \quad G^R(\omega) = G^A(\omega)^* = \frac{1}{P^R(\omega)} \tag{283}
\]

\[
G^K(\omega) = -G^R(\omega)P^K(\omega)P^A(\omega) = -\frac{2i\bar{\kappa}}{(\omega - \omega_0)^2 + \kappa^2} \tag{284}
\]

• structural properties (valid beyond explicit non-interacting example)

  – **Conservation of probability**: Zero matrix entry in Eq. (283), reflecting the zero in the inverse Green’s function in Eq. (281). This zero ensures \( S[\phi_c, \phi_q] = 0 \) for any \( \phi_c \) (recall the same property in MSR theory). Any Keldysh action has this property: Enforcing \( \phi_q = 0 \) is equivalent to setting \( \phi^+ = \phi^- \), in which case the action must vanish. This operation on the action acts as taking the trace in the quantum master equation: Using cyclic property under the trace we can move all operators on one side of \( \rho \) to see \( \partial_t \text{tr} \rho = 0 \), i.e., conservation of probability. This property holds not only for the bare action in the exponent of \( Z \), but also for the full effective action (see exercises)

  – Behavior under hermitian conjugation:

    \[
    G^R(\omega) = G^A(\omega), G^K(\omega) = -G^K(\omega) \tag{285}
    \]

    This is an exact property: can be checked on Eq. (276)

  – **Analytic structure**: the poles of \( G^{R(A)}(\omega) \) are in the lower (upper) half of the complex plane at \( \omega = \omega_0 \pm i\kappa \). Thus \( G^{R(A)} \) describe the retarded (advanced) response of a system perturbed at \( t = 0 \):

    \[
    G^R(t) = -i\theta(t)e^{-(i\omega_0 + \kappa)t} \tag{286}
    \]

    This is an exact property, holds for renormalized Green’s functions as well

  – **Connection to operator formalism**: we use:

    (i) The discrete field index holds information on the side of the density matrix on which the operator acts and

    (ii) functional integrals generate time ordered (+contour) or anti-time ordered (-contour) correlation functions:
\[ G^K(t, t') = -i \langle a_c(t) a^*_c(t') \rangle \]
\[ = -\frac{i}{2} \langle (a_+(t) + a_-(t))(a^*_+(t') + a^*_-(t')) \rangle \]
\[ = -\frac{i}{2} \left( \langle Ta(t)a^1(t') \rangle + \langle \{a(t), a^1(t')\} \rangle + \langle T a(t) a^1(t') \rangle \right) \]
\[ = -i \langle \{a(t), a^1(t')\} \rangle \] (287)

\[ T(\bar{T}) \text{ stand for time (anti-time) ordering; similarly,} \]
\[ G^R(t, t') = -i \theta(t - t') \langle [a(t), a^1(t')] \rangle \] (288)

– Physical interpretation: Correlation vs. response functions
  * two basic types of experiment:
    + correlation measurement: study without disturbing
      eg. quantum optics
    + response measurement: probe with (weak) external fields
      eg. transmission/absorption experiments

  photon output, eg. \( g^{(2)}(\tau) \)

  * Responses \( G^R/G^A \)
    - indeed \( G^R(t, t') \) describes how a system reacts at time \( t \) when perturbed at time \( t' \) (linear response)

\[
G^R(t, t') = -i \langle a_c(t) a^*_q(t') \rangle = \frac{\delta}{\delta j_q(t')} \langle a_c(t) \rangle \bigg|_{j_q=0} \]
\[ \text{damped cavity} = -i \theta(t - t') e^{-(i\omega_0 + \kappa)(t - t')} \] (289)

describes damped oscillations
- the retarded/advanced response contains information on the density of states (Lorentzian for this case)

\[ A(\omega) := -2 \text{Im} \ G^R(\omega) = - \left( G^R(\omega) - G^A(\omega) \right) \]
\[ \text{damped cavity} \frac{2\kappa}{(\omega - \omega_0)^2 + \kappa^2} \] (290)
It can be shown that $A(\omega) \geq 0$ in general. Moreover we have the sum rule

$$\int d\omega A(\omega) = \langle [a(t), a^\dagger(t)] \rangle = \langle [a, a^\dagger] \rangle = 1$$

(291)

i.e. $A(\omega)$ is a probability distribution, the probability to hit a resonance of the system.

* Correlations $G^K$:
  
  - $G^K$ contains information on how states are occupied. Instructive: equal time limit:

    $$G^K(0) = \langle \{a, a^\dagger\} \rangle = 2\langle a^\dagger a \rangle + 1$$

    $$= \int_0 d(\omega) \frac{2\kappa}{2\pi} \frac{1}{(\omega - \omega_0)^2 + \kappa^2} = \frac{\kappa}{\kappa} = 2\bar{n} + 1$$

    (292)

  where the last holds for a thermal bath, $\gamma_p = \gamma \bar{n}$, $\gamma_l = \gamma(\bar{n} + 1)$, i.e. $\langle a^\dagger a \rangle = \bar{n} \Rightarrow$ the cavity occupation equilibrates to the reservoir occupation

* Summary:
  
  Responses $G^{R/A} \leftrightarrow$ spectral information: which excitations are there?
  
  Correlation $G^K \leftrightarrow$ statistical information: how are states occupied?

• Final comment:

  - **stationary states**:

    * $t_0 \rightarrow -\infty, t_f \rightarrow +\infty$: **boundary value** problem, no memory of initial state.
      
      This is an assumption, similar to the assumption of thermal equilibrium

    * in stationary state, $G(t, t') = G(t - t')$ only (time translation invariance). Problems can be solved in frequency (and momentum $\leftrightarrow$ spatial translation invariance) domain, using QFT methods

  - **dynamical problems**
formalism can be adapted to initial value problems

strategy:
· derive equations of motion for correlation functions in time domain → typically non-closed hierarchy
· initial value = initial value of all correlation functions considered
· find suitable scheme to approximately decouple hierarchy to a closed subset

7.3 Semi-Classical limit

Q: Quantum systems behave classically in the presence of equilibrium noise = temperature. Under which circumstance does a driven quantum system behave classically, which simplifications apply?

Overview:

<table>
<thead>
<tr>
<th>QME</th>
<th>↔</th>
<th>Markovian Keldysh integral</th>
<th>→</th>
<th>MSR integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>↔ quantum Langevin equation</td>
<td>semiclassically limit</td>
<td>→ classical Langevin equation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>↔ quantum FPE</td>
<td></td>
<td>→ classical FPE</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Model (continuum)

- Hamiltonian:

\[
H = \int_x \hat{\phi}^\dagger(x) \left[ -\frac{\nabla^2}{2m} - \mu \right] \hat{\phi}(x) + \frac{g}{2} \left[ \left( \hat{\phi}^\dagger(x) \right)^2 \hat{\phi}(x)^2 \right]
\]

(293)

\[\text{Single point pump}\]

\[\text{MB system}\]

\[\text{Single/two-body law}\]

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• Lindblad operators

\[
    L_1(\vec{r}) = \hat{\phi}(\vec{r}) \quad \text{single particle loss: } \gamma_l; \quad L_3(\vec{r}) = \hat{\phi}^2(\vec{r}) \quad \text{two particle loss: } \kappa \tag{294}
\]

\[
    L_2(\vec{r}) = \hat{\phi}^\dagger(\vec{r}) \quad \text{single particle pump: } \gamma_p \tag{295}
\]

• action (Keldysh basis)

\[
    S = \int_{t,F} \left( \phi_c^* \frac{P^A}{2} \phi_c \right) \phi_q \phi_q^\dagger \phi_c^\dagger \phi_c + 4|\phi_c|^2 |\phi_q|^2 \tag{296}
\]

\[
    P^R = i\partial_t - \left( -\frac{\nabla^2}{2m} - \mu - i(\gamma_l - \gamma_p) \right) \tag{297}
\]

• EoM:

\[
    \frac{\delta S}{\delta \phi_q^\dagger} = 0 \Rightarrow i\partial_t \phi_c = \left( -\frac{\nabla^2}{2m} - \mu - i(\gamma_l - \gamma_p) + (g - i\kappa)|\phi_c|^2 \right) \phi_c \tag{298}
\]

\[+ i \left( (\gamma_l + \gamma_p) + 2(g - i\kappa)|\phi_q|^2 \phi_c + 2\kappa|\phi_c|^2 \right) \phi_q \]

• get orientation:

  – neglect noise (damping only), neglect spatial dependence:

\[
    \partial_t \phi_c = \left[ i\mu + (\gamma_p - \gamma_l) - (ig + \kappa)|\phi_c|^2 \right] \phi_c \tag{299}
\]

\[\rightarrow \text{overdamped motion in potential landscape shown in figure}\]

  – real part of Eq. (299): the steady state for \( \phi_c \) (denoted by \( \phi_0 \)) is determined by the flux equilibrium between effective pumping \( \gamma_p - \gamma_l \) and two-body loss:

\[
    \Rightarrow |\phi_0|^2 = \frac{\gamma_p - \gamma_l}{\kappa} \quad \text{for } \gamma_p - \gamma_l > 0 \tag{300}
\]

  – imaginary part: \( \mu \) is a free parameter which determines a rotating frame. It can be chosen to remove the time dependence of \( \phi_c \) in stationary state (\( \mu = g|\phi_0|^2 \)). It does \textbf{not} play the role of a chemical potential: as we have seen, the condensate density is fixed by the real part of Eq. (299)!
stationary state $\phi_0 = 0$

$|\phi_0|^2 = \frac{\pi - \gamma}{\kappa}$ (spontaneous symmetry breaking)

- we show the classical limit emerges whenever we move close to a critical point in generic driven dissipative quantum system

- tool: canonical power counting (strictly valid at a critical point, useful for $\omega \ll \mu$ (noise level))

  - Gaussian action:

    $$\int_{t, \vec{r}} \left( \phi_c^*, \phi_q^* \right) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix}$$

    (301)

  - frequency/momentum domain: $P^R = \omega - \left( \frac{\vec{q}^2}{2m} - \mu - i(\gamma_l - \gamma_p) \right) \rightarrow 0$ critical point

    Wlog (phase rotation)

    * $P^R \sim k^2 \leftarrow$ arbitrary momentum scale. (NB: $\omega = \frac{q^2}{2m} \Rightarrow \omega \sim k^2$)

    * but crucially $P^K = i(\gamma_l + \gamma_p) \sim k^0$

  - the scaling dimension of the action must vanish, $[S] = 0$

    $\Rightarrow [\phi_c] = \frac{d - 2}{2}$; $[\phi_q] = \frac{d + 2}{2}$ (302)

    $\Rightarrow$ canonical scaling of classical MSR functional reproduced!

  - from this: scaling dimension of vertices

    eg. $\int_{t, \vec{r}} \lambda_c |\phi_c|^2 \phi_c \phi_q$

    $[\lambda_c] = 4 - d > 0$ for $d < 4$

    relevant

    $[\lambda_q] = -d < 0$

    irrelevant

    $\Rightarrow$ massive simplifications ($d > 2$)
- R/A sector (odd in $\phi_q$): only linear term in $\phi_q$
- K sector (even in $\phi_q$): only quadratic noise term non-irrelevant

- the action in the semiclassical limit can thus be written in the form

$$S_c = \int_{t,r} \left[ \phi_q^* \frac{\delta \bar{S}[\phi_c]}{\delta \phi_c^*} + \text{h.c.} + 2i(\gamma_l + \gamma_p)|\phi_q|^2 \right]$$

where $\bar{S}$ is typically nonlinear in $\phi_c, \phi_c^*$. Its precise form is obtained from the microscopic action $S[\phi_+, \phi_-] = S[\sqrt{\frac{1}{2}}(\phi_c + \phi_q), \sqrt{\frac{1}{2}}(\phi_c - \phi_q)]$ expanded to second order in $\phi_q$.

- explicit expression including non-irrelevant terms in $2 < d < 4$:

$$S = \int_{t,r} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & i(\gamma_l + \gamma_p) \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} - \frac{1}{2}(g - i\kappa)|\phi_c|^2 \phi_c \phi_q^* + \text{c.c.} \right\}, \quad (304)$$

$$P^R = i\partial_t - \left(-\frac{\nabla^2}{2m} - \mu - i(\gamma_l - \gamma_p)\right)$$

7.4 Symmetries in the Keldysh formalism

- One key advantage of the functional integral formulation is the straightforward application of symmetry considerations to non-equilibrium problems. We focus on the following aspects:

7.4.1 Global symmetries: Noether & Goldstone theorems

- Just as there is a doubling of field variables, also symmetry generators may get doubled. As for the fields, there are "classical" and "quantum" symmetries

- The quantum symmetry is associated to conserved charges according to the Noether construction

- The classical symmetry can be broken spontaneously, in which case it gives rise to observable Goldstone modes

- The quantum symmetry is generically broken spontaneously in low frequency theories, but much less is known about this case, on which we only comment briefly
Noether theorem

- we consider actions $S = \int dt d^4r \mathcal{L}[\Phi, \partial_\mu \Phi]$ with local Lagrange densities $\mathcal{L}$ (depending on the field and its first derivative only)

- **continuous global** symmetry $\left( \Phi = \left( \phi_+, \phi_-, \phi_+^*, \phi_-^* \right)^T \right)$

  $$T_\alpha \Phi = \Phi + \alpha G \Phi + \mathcal{O}(\alpha^2) \quad (306)$$

  with $T_{\alpha=0} = 1$ and $G$ a $4 \times 4$ matrix in the Keldysh and Nambu spinor space with, in general, derivative operators as entries. $\alpha$ is $t, \vec{r}$-independent for global symmetries

- Symmetry property:

  $$S[T_\alpha \Phi] = S[\Phi], \quad \mathcal{D}[T_\alpha \Phi] = \mathcal{D}\Phi \quad (307)$$

- Implication I: for global $\alpha$, expansion of Eq. (307) gives the constraint

  $$\alpha \int dt d^4r \left( \frac{\partial \mathcal{L}}{\partial \Phi^T} G \Phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^T} \partial_\mu G \Phi \right) \equiv 0 \quad (308)$$

  for arbitrary $\alpha$. Thus, the integrand can be written as

  $$\partial_\mu f^\mu = \frac{\partial \mathcal{L}}{\partial \Phi^T} G \Phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^T} \partial_\mu G \Phi \quad (309)$$

- Implication II: we now study local transformations $\alpha(t, \vec{r})$. We assume that the functional measure is invariant under this transformation (this is the case for unitary $T_\alpha$):

  $$Z = \int \mathcal{D}\Phi e^{iS[\Phi]} = \int \mathcal{D}T_\alpha \Phi e^{iS[T_\alpha \Phi]} = \int \mathcal{D}\Phi e^{iS[T_\alpha \Phi]} \quad (310)$$

  where

  $$S[T_\alpha \Phi] = S[\Phi] + \int dt d^4r \left( \alpha \partial_\mu \left[ f^\mu - \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^T} G \Phi \right] \right) + \mathcal{O}(\alpha^2) \quad (311)$$

  i.e. there is one extra term compared to the global case

- inserting the latter into Eq. (310) and expanding in $\alpha$, we find (for any $\alpha$):

  $$\int dt d^4r \alpha \partial_\mu \left( f^\mu - \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^T} G \Phi \right) \quad (312)$$
we thus obtain the **Noether current** obeying the continuity equation

\[ j^\mu = \frac{\partial L}{\partial (\partial_\mu \Phi^T)} G\Phi - f^\mu, \quad \partial_\mu (j^\mu) = 0 \]  (313)

\[ j^\mu \] not unique; for \( a, b^\mu \) constant, \( aj^\mu + b^\mu \) is also a conserved current

- The zero component is the conserved **Noether charge**:

\[ Q = \int d^4r j^0 = \int d^4r \left[ \frac{\partial L}{\partial (\partial_0 \Phi^T)} G\Phi - f^0 \right] \]  (314)

- **Application:** phase rotation symmetry \( U(1) \)

  - Lindblad equation with "dephasing":

\[ \partial_t \rho = -i \int d^4r [\hat{H}(\vec{r}), \rho] + \kappa \int d^4r \left( \hat{L}(\vec{r}) \rho \hat{L}^\dagger(\vec{r}) - \frac{1}{2} \left\{ \hat{L}^\dagger(\vec{r}) \hat{L}(\vec{r}), \rho \right\} \right) \]  (315)

\[ \hat{H}(\vec{r}) = \hat{\phi}^\dagger(\vec{r}) \frac{-\Delta}{2m} \hat{\phi}(\vec{r}) + \frac{\lambda}{2} \hat{\phi}^2(\vec{r}) \hat{\phi}^2(\vec{r}) \]  (316)

\[ \hat{L}(\vec{r}) = \hat{\phi}^\dagger(\vec{r}) \hat{\phi}(\vec{r}) = \hat{n}(\vec{r}) \text{ local density} \]  (317)

  - number conservation \( [\hat{H}(\vec{r}), \hat{N}] = [\hat{L}(\vec{r}), \hat{N}] = 0 \forall \vec{r}, \hat{N} = \int d^4r \hat{n}(\vec{r}) \)

  - goal: reproduce this result from Keldysh field theory; obtain the local Noether current; learn about the structure of symmetries

  - Keldysh action:

\[ S = \int dt d^4r \left( \phi_+^*(t) i \partial_t \phi_+ - \phi_-^* i \partial_t \phi_- (t) + \mathcal{H}_+ - \mathcal{H}_- - i \left( L_+ L_-^* - \frac{1}{2} L_+^* L_+ - \frac{1}{2} L_- L_-^* \right) \right) \]  (318)

with \( \mathcal{H}_\sigma = \mathcal{H}[\phi_\sigma] \) etc. (we ignore normal ordering issues that are present in this problem)

  - note: we have two \( U(1) \) symmetries, as we can perform independent global phase rotations \( \alpha_+, \alpha_- \) on each contour without changing the action,

\[ \phi_\sigma(\vec{r}) \rightarrow e^{i\alpha_\sigma} \phi_\sigma(\vec{r}), \quad \phi_\sigma^*(\vec{r}) \rightarrow e^{-i\alpha_\sigma} \phi_\sigma^*(\vec{r}) \]  (319)

  - these independent transformations can be realized by \( \alpha_c = \frac{1}{2} (\alpha_+ + \alpha_-) \) ("classical" transformation \( U_c(1) \)) and \( \alpha_q = \frac{1}{2} (\alpha_+ - \alpha_-) \) ("quantum" transformation \( U_q(1) \))

  - In the language of Eq. (306), we can implement these by choosing
* classical transformation $\alpha_+ = \alpha$, $\alpha_- = \alpha$:

$$
\Rightarrow G_c = i \begin{pmatrix}
\sigma_z & 0 \\
0 & \sigma_z
\end{pmatrix}
$$

(320)

* quantum transformation $\alpha_+ = \alpha$, $\alpha_- = -\alpha$:

$$
\Rightarrow G_q = i \begin{pmatrix}
\sigma_z & 0 \\
0 & -\sigma_z
\end{pmatrix}
$$

(321)

– for convenience we define contour Noether currents by ($g = i\sigma_z$ here)

$$
j_\mu^c = \partial L / \partial (\partial_\mu \Phi^T_\sigma) g_{\Phi_\sigma} - f_\mu^c, \quad \partial_\mu f_\mu^c = \frac{\partial L}{\partial \Phi^T_\sigma} g_{\Phi_\sigma} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^T_\sigma)} \partial_\mu g_{\Phi_\sigma}
$$

(322)

– for our example:

* exactly invariant under global transformations $\Rightarrow \partial_\mu f_\mu^c \equiv 0$

* furthermore, the Lindblad term is strictly local (no derivatives)

$$
\Rightarrow j_\mu^c = \frac{\partial L}{\partial (\partial_\mu \Phi^T_\sigma)} g_{\Phi_c} = \sigma \frac{\partial \mathcal{H}_\sigma}{\partial (\partial_\mu \Phi^T_\sigma)} g_{\Phi_c}
$$

(323)

* accounting for the signs in Eqs. (320), (321), we then find:

quantum transformation $U_q(1)$: classical current $j_\mu^c = \frac{1}{2} \langle j_\mu^c + j_\mu^q \rangle$, (324)

classical transformation $U_c(1)$: quantum current $j_\mu^q = \langle j_\mu^c - j_\mu^q \rangle$ (325)

(we have used the freedom to premultiply a constant in $j_\mu^c$)

* This terminology is motivated by the following observation: $j_c$, $j_q$ are local operators (here: bilinears in the fields $\Phi_\sigma$), which transform as the classical and quantum fields under contour exchange ($\Phi_\sigma \rightarrow \Phi_{-\sigma}$): $j_\mu^c \rightarrow j_\mu^c$, $j_\mu^q \rightarrow -j_\mu^q$.

Therefore, and most importantly, for their expectation value we have $\langle j_\mu^c \rangle = 0$, while $\langle j_\mu^q \rangle$ can be nonzero:

quantum transformation $U_q(1)$: nontrivial charge $\int d^d r \langle j_\mu^q \rangle \neq 0$, (326)

classical transformation $U_c(1)$: trivial charge $\int d^d r \langle j_\mu^q \rangle = 0$ (327)
* for $U_q(1)$ we get explicitly for our microscopic model

$$\langle j_{\mu}^c \rangle = \frac{1}{2}(j_{\mu}^c + j_{\mu}^q),$$

$$j_{\sigma}^c = \left( \rho_{\sigma}, \vec{j}_{\sigma} \right)^T, \rho_{\sigma} = |\phi_{\sigma}|^2, \vec{j}_{\sigma} = \frac{1}{2im} (\phi^*_\sigma \nabla \phi_{\sigma} - \phi_{\sigma} \nabla \phi^*_\sigma)$$

and $N = \int d^d r \langle \phi^*_c(t, \vec{r}) \phi_c(t, \vec{r}) \rangle$ the total particle number expectation value as expected

- these statements can be straightforwardly generalized beyond $U(1)$ symmetry (cf. L. Sieberer, Reports on Progress in Physics 79, 096001 (2016))

**Goldstone theorem**

- the statement:

  - given a global continuous symmetry (see above)
  - spontaneous symmetry breaking $\langle \Phi \rangle \neq 0$ (at least non-vanishing component)

  $\Rightarrow$ There are gapless excitations *(Goldstone modes)*

- example: classical $|\phi^4|$ theory (see also potential landscape above)

  ![Goldstone Modes](image)

- identification of Goldstone modes: (i) promote $T_\alpha$ to a **local** symmetry (ii) integrate out all massive modes to get the Goldstone action

$$S[T_\alpha \Phi] = S[\Phi] + \Delta S[\partial_\mu \alpha, \Phi]$$

in such a way that $\Delta S[0, \Phi] = 0$ (i.e. for homogeneous $\alpha$): $\Delta S$ is a pure derivative action

- focus here on **classical** phase rotation symmetry $U_c(1)$:

$$T_\alpha \phi_c(t, \vec{r}) = e^{i\alpha} \phi_c(t, \vec{r}), \quad T_\alpha \phi_q(t, \vec{r}) = e^{i\alpha} \phi_q(t, \vec{r})$$
(both contours transform "in phase", thus there is a common transformation for $\phi_c, \phi_q$)

- this motivates the **phase-amplitude-parametrization**

$$
\phi_c(t, \vec{r}) = e^{i\theta(t, \vec{r})}\sqrt{\rho(t, \vec{r})}, \quad \phi_q(t, \vec{r}) = e^{i\theta(t, \vec{r})}\zeta(t, \vec{r})
$$

where the physical phase is identified with the Goldstone mode. $\rho$ is real, $\zeta$ is complex.

- we start from the model Eq. (296) in the semi-classical limit, Eq. (304), writing the partition function and action directly in phase-amplitude-representation:

$$
Z = \int \mathcal{D}[\phi_c, \phi_c^*, \phi_q, \phi_q^*] e^{iS} = \int \mathcal{D}[\rho, \theta, \zeta, \zeta^*] e^{iS}
$$

- The saddle point equations for $\rho$ and $\zeta$

$$
\frac{\delta S}{\delta \rho} = 0, \quad \frac{\delta S}{\delta \zeta} = 0
$$

precisely recover the mean field condition Eq. (300) for $|\phi_0|^2 = \rho_0$, $\rho_0$ the solution of (333)

- expand around this saddle point to second order in fluctuations ($\pi = \rho - \rho_0$) and for $\zeta = \zeta_1 + i\zeta_2$ as we can expect them to be generically gapped, but keep the full dependence on $\theta$, as this is the gapless Goldstone mode:

$$
S = 2 \int_{t, x} \left\{ \sqrt{\rho_0} \left\{ -\zeta_1 \left[ \partial_t \theta + K_c (\nabla \theta)^2 \right] + K_c \zeta_2 \nabla^2 \theta - (u_c \zeta_1 - u_d \zeta_2) \pi \right\} + i \left( \gamma + 2 u_d \rho_0 \right) |\zeta|^2 \right\}
$$

- this equation is linear in $\pi$, so we obtain a $\delta$-constraint

$$
Z = \int \mathcal{D}[\theta, \zeta_1, \zeta_2] \delta[u_c \zeta_1 - u_d \zeta_2] e^{iS'} = \int \mathcal{D}[\theta, \tilde{\theta}] e^{iS_{KPZ}},
$$

$$
S_{KPZ} = \int_{t, x} \tilde{\theta} \left[ \partial_t \theta - D \nabla^2 \theta - \frac{\lambda}{2} \left( \nabla \theta \right)^2 - \Delta \tilde{\theta} \right]
$$

$\rightarrow$ Interestingly, the Goldstone mode action of the driven-dissipative boson system is precisely the KPZ action studied in Sec. 5.2! (with effective noise field $\tilde{\theta} = 2i\sqrt{\rho_0}\zeta_1$)
the KPZ parameters are given as a function of microscopic parameters as

\[ D = K_c(r_u + 1/r_K), \quad \lambda = -2K_c(1 - r_u/r_K), \]

\[ \Delta = \frac{\gamma_l + \gamma_p}{\rho_0} (1 + r_u^2) \]

(337)

(338)

with ratios of coherent and dissipative couplings \( r_g = g_c/g_d \). We have added a diffusive term to the microscopic model, i.e. a contribution to the master equation

\[ \kappa_d \int d\vec{r} \left[ \nabla \phi(\vec{r}) \rho \nabla \phi^\dagger(\vec{r}) - \frac{1}{2} \{ \nabla \phi^\dagger(\vec{r}) \nabla \phi(\vec{r}), \rho \} \right] \]

(339)

Discussion:

– for equilibrium conditions, \( r_u = r_K \Rightarrow \lambda = 0 \). Indeed, applying the symmetry Eq. (357) in the following chapter adapted to real fields, the non-linearity \( \lambda \) must vanish under equilibrium conditions. Thus, \( \lambda \) is a direct measure of the strength of non-equilibrium conditions (cf. also the discussion of Sec. 5.2.1)

– In the usual KPZ equation, the deterministic dynamical variable is a growing interface, here it is a phase. The crucial difference lies in the fact that a phase is a compact variable (defined mod \( 2\pi \)), while an interface is non-compact. We will come back to this later.

– the precise form of the Goldstone excitation is fixed by the presence or absence of additional symmetries. Our microscopic model Eq. (296) only possesses a classical phase rotation symmetry \( U_c(1) \). Requiring additionally a quantum phase rotation symmetry, the nature of excitations changes:

\[ \text{only } U_c(1) : \quad \omega \simeq iDq^2 \quad \text{diffusive Goldstone mode} \]

\[ U_c(1) \& U_q(1) : \quad \omega \simeq c|\vec{q}| \quad \text{ballistic (sound) Goldstone mode} \]

For a quick (but incomplete) way to see the second equation, discard all dissipative terms in Eq. (296), recovering a closed Bose gas at \( T = 0 \) upon performing a Gross-Pitaevski approximation). Then, number conservation is restored and a sound mode results. A more complete picture at finite \( T \) is obtained from ” Model E” of Halperin and Hohenberg, where additional slow hydrodynamic modes need to be considered to produce the expected sound mode

– Breaking of \( U_q(1) \) symmetry:
7.4.2 Thermodynamic equilibrium as a symmetry of the Keldysh action

- goal: show that the conditions of thermodynamic equilibrium, measured by the presence of thermal quantum fluctuation-dissipation relations, can be encoded in a single symmetry transformation that is present for any dynamics generated by a time-independent Hamiltonian operator alone

Preparations

- thermodynamic equilibrium at temperature $T = 1/\beta$ is defined by two conditions:
  
  (i) the density matrix is given by $\rho = e^{-\beta H}$, $H$ time independent
  
  (ii) the very same $H$ generates the dynamics of the system

- note that (ii) is crucial: any $\rho$ (positive semidefinite hermitean matrix with unit trace) can be parametrized as $\rho = e^{-\beta H}$ with $H$ hermitian. In other words, dynamics is in general crucial to decide on thermal equilibrium conditions; the static correlations obtained from $\rho$ are not sufficient in general.

- Implications on observables: any correlator of Heisenberg operators $A(t), B(t)$ obeys the KMS condition:

  \[ \langle A(t)B(t') \rangle = \langle B(t')A(t + i\beta) \rangle = \langle B(t' - i\beta/2)A(t + i\beta/2) \rangle \]  

proof: cyclic invariance and (i), (ii):

\[ \rho A(t) = e^{-\beta H} e^{iHt} A e^{-iHt} = e^{i(t+i\beta)H} A e^{-(t+i\beta)H} e^{-\beta H} \]

\[ = A(t + i\beta)\rho \]  

- Implications: Fluctuation-dissipation relations (FDRs). Eg. accounting for the time orderings, after Fourier transform one sees (bosons $\phi, \phi^\dagger$)

\[ G^K(\omega, \vec{q}) = \coth \frac{\omega}{2T} \left( G^R(\omega, \vec{q}) - G^A(\omega, \vec{q}) \right) \]
relation between single-particle correlation and response functions governed by the Bose distribution \( \text{coth} \frac{\omega}{2T} = 2n_B(\omega) + 1, \ n_B(\omega) = (\exp \frac{\omega}{T} - 1)^{-1} \)

higher order FDRs: "straightforward", but very tedious

**Intuition:** classical limit. Implement by restoring units, \( \omega/2T \to \frac{\hbar \omega}{2k_B T} \), and consider \( \frac{\hbar \omega}{2k_B T} \ll 1 \), i.e. either \( \hbar \to 0 \) or \( T \to \infty \). Then \( \text{coth} \frac{\omega}{2T} \to \frac{2T}{\omega} \)

\[ \Rightarrow \frac{\omega}{2T} G^K(\omega) = G^R(\omega) - G^A(\omega) \quad (343) \]

\[ \Rightarrow \frac{1}{2T} i \partial_t G^K(t) = G^R(t) - G^A(t) \ | \ \theta(t) . \quad (344) \]

\[ \Rightarrow \frac{i}{2T} \theta(t) \partial_t G^K(t) = G^R(t) \quad (345) \]

\[ \Rightarrow \frac{\delta(\phi_c(t))}{\delta j_c(0)} = \frac{1}{2T} \theta(t) \partial_t \langle \phi_c(t) \phi_c^*(0) \rangle \quad (346) \]

inducing a response of the system by varying the source is connected to the temporal change of correlation functions with proportionality factor set by temperature. Quantum case: not local in time, less intuitive.

**Discussion:** to test for thermodynamic equilibrium operationally, one thus has to test an infinite hierarchy of correlation functions (KMS relation or FDRs). We will take another route here and encode the conditions of thermodynamic equilibrium in terms of a symmetry of the microscopic Keldysh action. Using the concept of the effective action (see QFT 1), this symmetry can then also be tested for the full effective action, making the test of infinitely many FDR’s unnecessary. Conversely, the symmetry’s absence immediately indicates non-equilibrium conditions.

**The symmetry**

**The transformation** (L. Sieberer et al., PRB 92, 134 307 (2015)), discarding spatial arguments for notational ease:

- time and frequency domains, ± basis

\[
\mathcal{T}_\beta \phi_\sigma(t) = \phi_\sigma^*(-t + i\sigma \beta/2) \quad \mathcal{T}_\beta \phi_\sigma(\omega) = e^{-i\beta \omega/2} \phi_\sigma^*(\omega) \\
\mathcal{T}_\beta \phi_\sigma^*(t) = \phi_\sigma(-t + i\sigma \beta/2) \quad \mathcal{T}_\beta \phi_\sigma^*(\omega) = e^{i\beta \omega/2} \phi_\sigma(\omega) \quad (347)
\]
– frequency domain, RAK basis

\[ T_\beta \phi_c(\omega) = \cosh \beta \omega/2 \phi^*_c(\omega) - \sinh \beta \omega/2 \phi^*_q(\omega) \]

\[ T_\beta \phi_q(\omega) = -\sinh \beta \omega/2 \phi^*_c(\omega) + \cosh \beta \omega/2 \phi^*_q(\omega) \] (348)

– including chemical potential:

\[ T_\beta \phi_\sigma(t) = e^{\sigma \beta \mu/2} \phi^*_\sigma(-t + i\sigma \beta/2) \]

\[ T_\beta \phi_\sigma^*(t) = e^{-\sigma \beta \mu/2} \phi_\sigma(-t + i\sigma \beta/2) \]

\[ T_\beta e^{\sigma \beta \mu/2} \phi_\sigma^*(\omega) = e^{\sigma \beta \mu/2}e^{\sigma \beta \omega/2} \phi_\sigma(\omega) = \phi_\sigma(\omega) \] for any \( \phi_\sigma(\omega) \) (349)

• properties:

– composition of:

* time reversal (complex conjugation of fields, inversion of sign of \( t \))

* translation of \( t \) by \( i\sigma \beta/2 \) (cf. KMS in symmetrized version Eq. (340); cf. infinitesimal real translation of \( t \) has energy as Noether charge)

– the transformation is involutory \( T_\beta^2 = 1 \) and therefore discrete:

\[ T_\beta^2 \phi_\sigma(\omega) = T_\beta e^{-\sigma \beta \mu/2} \phi^*_\sigma(\omega) = e^{-\sigma \beta \mu/2}e^{\sigma \beta \omega/2} \phi_\sigma(\omega) = \phi_\sigma(\omega) \] (350)

(this is different from the real infinitesimal time translations)

• key point: the action of any time-independent, local Hamiltonian possesses this symmetry. Eg. for any local field monomial,

\[ T_\beta \sum_\sigma \int_t \sigma (\phi^*_\sigma(t)\phi_\sigma(t))^n \]

\[ = \int_t \sigma (\phi_\sigma(-t + i\beta/2)\phi^*_\sigma(-t + i\beta/2))^n = \int_t \sigma (\phi^*_\sigma(t)\phi_\sigma(t))^n \] (351)

shifting the integration contour and inverting the time direction under the integral. (More clean derivation in frequency domain). Works for any monomial with gradients. In the presence of external fields breaking time reversal explicitly (magnetic fields), extensions are necessary but possible

• however, the Keldysh functional integral for such a purely Hamiltonian action is not well
defined:

\[ Z = \int D(\Phi_+, \Phi_-) e^{iS[\Phi_+, \Phi_-]} , \quad S = \int dt \left( \phi_+^* i\partial_t \phi_+ - \phi_-^* i\partial_t \phi_- + (H[\Phi_+] - H[\Phi_-]) \right) \] (352)

without imaginary parts (which we had in the Lindblad equation), the integral is purely oscillatory and does not converge

- we fix this by introducing imaginary parts which are compatible with the symmetry: regularization of the Keldysh field theory. In the single particle sector (quadratic part of the action), the most general structure takes the form (RAK basis)

\[ S_{\text{reg}} = \int_\omega \begin{pmatrix} \phi_+^*(\omega), \phi_-^*(\omega) \end{pmatrix} \begin{pmatrix} 0 & -i\epsilon g(\omega) \\ i\epsilon g(\omega) & 2i\epsilon g(\omega) \coth \frac{\omega}{2T} \end{pmatrix} \begin{pmatrix} \phi_c(\omega) \\ \phi_q(\omega) \end{pmatrix} \] (353)

- with arbitrary real \( g(\omega) \). For a regularization, we take \( \epsilon \to 0 \) real infinitesimal
- the Bose distribution emerges naturally from the requirement of \( T_\beta \) invariance!

- Implication: FDRs of arbitrary order as Ward identities of the symmetry

- strategy to generate Ward identities (symmetries in general):

\[ \text{symmetry: } S[\Phi] = S[T_\beta \Phi], \quad D(\Phi) = D(T_\beta \Phi) \] (354)

(the latter can be checked explicitly: the Jacobian of \( T_\beta \) is unity)

- thus:

\[ \langle \phi_\sigma(t)\phi_\sigma^*(t') \rangle = \int D\Phi \phi_\sigma(t)\phi_\sigma^*(t')e^{iS[\Phi]} = \int D(T_\beta \Phi) T_\beta[\phi_\sigma(t)\phi_\sigma^*(t')]e^{iS[T_\beta \Phi]} = \int D\Phi T_\beta[\phi_\sigma(t)\phi_\sigma^*(t')]e^{iS[\Phi]} = \langle T_\beta \phi_\sigma(t), T_\beta \phi_\sigma^*(t') \rangle \] (355)

- example: single particle FDR. Consider above identity for

\[ 0 = \langle \phi_\sigma(\omega)\phi_\sigma^*(\omega') \rangle = \langle T_\beta \phi_\sigma(\omega)T_\beta \phi_\sigma^*(\omega') \rangle \] (356)

\( T_\beta \) mixes \( q \) and \( c \) components. This yields (using that all appearing correlators are \( \sim \delta(\omega - \omega') \))

\[ \langle \phi_\sigma^*(\omega)\phi_\sigma^*(\omega') \rangle = \coth \frac{\omega}{2T} \left( \langle \phi_\sigma(\omega)\phi_\sigma^*(\omega') \rangle - \langle \phi_\sigma(\omega)\phi_\sigma^*(\omega') \rangle \right) \] (357)
i.e. Eq. (342)

- example: more precise notion of "time independent Hamiltonian"

- action: single bosonic mode with frequency $\omega_0$, driven with frequency $\omega_d$ and amplitude $\Omega$, and coupled to a bath of harmonic oscillators:

\[
S = S_0 + S_d + S_b + S_{sb}
\]

\[
S_0 = \sum_{\sigma} \int \sigma \left( a_\sigma^*(\omega) (\omega - \omega_0) a_\sigma(\omega) \right)
\]

\[
S_d = \Omega \sum_{\sigma} \int dt \left( a_\sigma(t) e^{i\omega_d t} + a_\sigma^*(t) e^{-i\omega_d t} \right) = \Omega \sum_{\sigma} \left( a_\sigma(\omega_d) + a_\sigma^*(\omega_d) \right)
\]

\[
S_b = \sum_{\mu} \int \sigma \left( b_{\mu,c}^*(\omega), b_{\mu,d}^*(\omega) \right) \times \left( \begin{array}{cc} 0 & \omega - \omega_\mu - i\delta \\ \omega - \omega_\mu + i\delta & 2\delta \coth(\beta\omega/2) \end{array} \right) \left( b_{\mu,c}(\omega), b_{\mu,d}(\omega) \right)
\]

\[
S_{sb} = \lambda \sum_{\sigma} \sum_{\mu} \int \sigma \left( a_\sigma^*(\omega) b_{\mu,\sigma}(\omega) + a_\sigma(\omega) b_{\mu,\sigma}^*(\omega) \right)
\]

- $S_b$ and $S_{sb}$ are invariant under Eq. (349) with $\mu = 0$. The driving part violates the symmetry

\[
S_d = \Omega \sum_{\sigma} \left( a_\sigma(\omega_d) e^{\sigma\beta\omega_d/2} + a_\sigma^*(\omega_d) e^{-\sigma\beta\omega_d/2} \right)
\]

- can we restore the symmetry by introducing a rotating frame $\tilde{a}_\sigma(t) = a_\sigma(t) e^{i\omega_d t}, \tilde{b}_\sigma(t) = b_\sigma(t) e^{i\omega_d t}$? Then $S_d$ transforms into

\[
\tilde{S}_d = \Omega \sum_{\sigma} \int dt \left( \tilde{a}_\sigma(t) + \tilde{a}_\sigma^*(t) \right) = \Omega \sum_{\sigma} \left( \tilde{a}_\sigma(0) + \tilde{a}_\sigma^*(0) \right)
\]

- Now this term, $S_0$ and $S_{sb}$ are invariant, but $S_b$ acquires an effective chemical potential $\coth(\beta\omega/2) \rightarrow \coth(\beta(\omega - \omega_d)/2)$

$\Rightarrow$ key point: **No rotating frame exists in which the total action is invariant.** This statement is more precise for $T_\beta$ to be a symmetry than saying that the microscopic Hamiltonian is time independent

**Classical limit**

- starting point: apply the canonical power counting of Eq. (302) to the symmetry trans-
formation in Eq. (348):

\[
T_\beta \phi_c(t) = \phi_c^*(-t)
\]

\[
T_\beta \phi_q(t) = \sigma_x \left( \phi_q^*(-t) + \frac{i}{2T} \partial_t \phi_c^*(-t) \right)
\]  

(365)

• implications:

1. the classical FDR (see Eq. (345)) is recovered as Ward identity

\[
G^k(\omega) = \frac{2T}{\omega} \left( G^R(\omega) - G^A(\omega) \right)
\]

2. geometric constraint on the location of couplings in complex plane:
   
   – action in (semi-)classical limit ("semi": effects of phase coherence still present)

\[
S = \int_{t,\vec{r}} \left( \phi_q^* \left[ i \partial_t \phi_c - \left( \frac{\delta H_c}{\delta \phi_c^*} - i \frac{\delta H_d}{\delta \phi_c^*} \right) \right] + 2iT \phi_q^* \phi_q \right)
\]

(366)

\[
H_\alpha = \int_{t,\vec{r}} \left( K_\alpha |\nabla \phi_c|^2 + r_\alpha |\phi_c|^2 + \frac{u_\alpha}{2} |\phi_c|^4 \right)
\]

(367)

\(H_\alpha\) are Landau-Ginzburg functionals that parametrize reversible (\(\alpha = c\)) and irreversible (\(\alpha = d\)) dynamics

– The symmetry (365) is present precisely for the case

\[
H_c = r H_d \quad \Leftrightarrow \quad r = \frac{K_c}{K_d} = \frac{u_c}{u_d} = \ldots
\]

(368)

– visualization: plot the real \((H_c)\) and imaginary \((H_d)\) parts of couplings

• discussion:
- equilibrium: all couplings aligned; non-equilibrium: couplings spread in complex plane

- intuition: in thermal equilibrium, reversible and irreversible dynamics are not independent, but originate from one single dynamical resource (the microscopic Hamiltonian). Fig. (a) can be understood as a coarse grained Hamilton dynamics starting microscopically on the real axis. In contrast, in a non-equilibrium system, there are independent dynamical resources for reversible and irreversible dynamics, which causes spread in the complex plane (Fig (b))

- interestingly, in the classical limit, this discussion shows that equilibrium and non-equilibrium situations can be distinguished based on static observables (technically, the one-particle irreducible scattering vertices). In contrast, the FDR’s are based on dynamic (finite frequency) information
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