Universität zu Köln
Institut für Theoretische Physik
Prof. Dr. Claus Kiefer
Carsten Weber

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Exercise 2 of Theoretische Physik II: Elektrodynamik
Green's functions, distributions

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Problem 1 (3 points): *some more Fourier analysis*

1. Prove the following properties of the Fourier transformation: ($\tilde{f} = \mathcal{F}(f)$)

   (i) $\tilde{\tilde{f}}(t) = f(-t)$

   (ii) $\langle f, g\rangle = \langle \tilde{f}, \tilde{g}\rangle$

   (iii) $\mathcal{F}(f * g) = \tilde{f}\,\tilde{g}$

(3 points)

Problem 2 (6 points): *distributions*

1. Show the following properties of the δ-distribution (by means of test functions $\varphi \in \mathscr{S}$):

   (i) $\delta(at) = \frac{1}{|a|}\delta(t)$
   (ii) $\delta(g(t)) = \sum_i \frac{1}{|g'(t_i)|}\delta(t - t_i)$, t_i zero of $g(t)$

(2 points)

2. Consider the functions

$$f_n(t) = \frac{1}{\pi}\frac{n}{1 + n^2 t^2}.$$

Show that in the limit of $n \to \infty$, $f_n(t)$ converges to $\delta(t)$.

(2 points)

3. Prove by letting the expressions act on test functions $\varphi \in \mathscr{S}$:

   (i) $f(t)\delta(t) = f(0)\delta(t)$
   (ii) $f(t)\delta'(t) = f(0)\delta'(t) - f'(t)\delta(t)$,

where $f \in \mathscr{S}$. (2 points)

Problem 3 (10 points): *Green's functions*

The goal of this problem is to derive the retarded Green's function G_n of the wave equation in n spatial dimensions (throughout this problem, we set c=1)

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_n\right) G_n(t, x) = \delta(t)\delta(x) \qquad (1)$$

where $G_n(t < 0, x) = 0$ and the spatial derivatives vanish in spatial infinity. Let $x = (x_1, ..., x_n)$ and

$$\Delta_n = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}.$$

Make the ansatz $G_n(t, x) = g_n(t, r)$ where $r = \sqrt{x_1^2 + ... + x_n^2}$.

1. Show that

$$G_n(t, x) = \int_{-\infty}^{\infty} dx_{n+1}\, G_{n+1}(t, x, x_{n+1})$$

is a Green's function according to (1) in n spatial dimensions, if G_{n+1} is also one in $n+1$ spatial dimensions. *(reduction method of Hadamard)* (1 points)

2. Show by direct calculation that

$$g_1(t, r) = \frac{1}{2}\Theta(t - r), \quad g_3(t, r) = \frac{1}{4\pi}\frac{\delta(t - r)}{r}$$

and, from this, calculate $g_2(t, r)$.
(Use the fact that the rotationally symmetric representation of the derivative operators in spherical coordinates are given by $\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$, $\nabla\delta(t - r) = \frac{\partial}{\partial r}\delta(t - r)\frac{\mathbf{x}}{r}$, as well as $\Delta\frac{1}{r} = -4\pi\delta(x)$ and 2.1(ii).) (4 points)

3. By applying the reduction method twice in a row, write $g_n(t, r)$ as an area integral over an expression which includes g_{n+2}. Use polar coordinates for the area integral to derive the recursion formula

$$g_{n+2}(t, r) = -\frac{1}{2\pi}\frac{1}{r}\frac{\partial}{\partial r} g_n(t, r)$$

(2 points)

4. Give the exact formulas for g_{2k} and g_{2k+1}. The expressions can still contain derivative operators. From this, calculate g_4 and g_5. (3 points)

5. Physically, the Green's function here is the temporal and spatial evolution of a (δ-) light pulse which is emitted at time $t = 0$ at the place $x = 0$. Which signal (qualitatively) does an observer see for $t \geq r$ for even and odd spatial dimensions? (It can be shown that generally for even spatial dimensions, the observer would register a "reverberation"; this effect is not present for odd spatial dimensions.)

(1 point)