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Exercise 6 of Theoretische Physik II: Elektrodynamik

multipole expansion, magnetostatics

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Problem 1 (7 points): *electrostatic multipole expansion*

1. In the lecture, the multipole expansion was derived in spherical coordinates. Now carry out this derivation in Cartesian coordinates. To this end, expand the electrostatic potential

$$\Phi(\mathbf{x}) = k \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

into a Taylor series up to second order (corresponding to the quadrupole moment). Bring the expansion into the form

$$\Phi(\mathbf{x}) = k \left(\frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{x}}{r^3} + \frac{1}{2} \frac{\mathbf{x}^{\mathrm{T}} \mathbf{Q}\, \mathbf{x}}{r^5} \right)$$

by defining the quadrupole moment $\mathbf{Q}$ in such a way that it is tracefree ($Q_{ii} = 0$). (3 points)

2. Consider an ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + \left(\frac{z}{b}\right)^2 \leq 1$$

with homogeneously distributed total charge Q and half-axes a, a, and b. Calculate the quadrupole moment and determine the potential in quadrupole approximation. (Hint: you may use exercise 5, problem 2.) In this approximation, consider the case $\left(\frac{b}{a}\right) = 1 + \epsilon, |\epsilon| \ll 1$ and discuss the result! (4 points)

Problem 2 (8 points): *magnetostatics*

A homogeneously charged spherical shell of total charge Q is rotating at constant angular velocity ω around the z-axis.

1. What is the current density $\mathbf{j}(\mathbf{x})$? (1 point)

2. Calculate the magnetic dipole moment. (2 points)

3. Determine the vector potential $\mathbf{A}(\mathbf{x})$ and the magnetic induction $\mathbf{B}(\mathbf{x})$ everywhere. (5 points)

(Hint: $\int\limits_{-1}^{1} \xi(c-\xi)^{-1/2} \mathrm{d}\xi = \frac{4}{3}\mathrm{c}(\sqrt{\mathrm{c}+1} - \sqrt{\mathrm{c}-1}) - \frac{2}{3}(\sqrt{\mathrm{c}+1} + \sqrt{\mathrm{c}-1})$.)

Problem 3 (5 points): *solenoid*

Consider an (infinitely long) solenoid with N_S windings per length l_S on a circular cylinder with radius R, through which a constant current I is flowing.

1. What is the current density $\mathbf{j}(\mathbf{x})$? (1 point)

2. Determine the vector potential $\mathbf{A}(\mathbf{x})$ and the magnetic induction $\mathbf{B}(\mathbf{x})$ within and outside of the solenoid. Make an appropriate ansatz for the vector potential and use Ampère's law. Using the boundary and continuity conditions, one can calculate $\mathbf{A}(\mathbf{x})$ and thus $\mathbf{B}(\mathbf{x})$. (4 points)