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Exercise 9 of Theoretische Physik II: Elektrodynamik
Liénard-Wiechert potentials, dielectrics

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Problem 1 (9 points): *Liénard-Wiechert potentials*

In this exercise we consider the potentials of moving charges, so-called Liénard-Wiechert potentials.

1. Starting from the general form of the retarded potentials

$$u(\mathbf{x},t) = \frac{1}{4\pi}\int d^3x' \int dt' \, \frac{h(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\,\delta(t-t'-\frac{|\mathbf{x}-\mathbf{x}'|}{c}),$$

where $h(\mathbf{x},t) = \begin{cases} \frac{1}{\epsilon_0}\rho(\mathbf{x},t) & : \; u(\mathbf{x},t)=\phi(\mathbf{x},t) \\ \mu_0 j_i(\mathbf{x},t) & : \; u(\mathbf{x},t) = A_i(\mathbf{x},t) \end{cases}$, show that the potentials for a moving point charge q with charge density $\rho(\mathbf{x},t) = q\delta(\mathbf{x}-\mathbf{X}(t))$ and current density $\mathbf{j}(\mathbf{x},t) = q\mathbf{V}(t)\delta(\mathbf{x}-\mathbf{X}(t))$ are given by

$$\phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R(t') - \frac{\mathbf{V}(t')\cdot\mathbf{R}(t')}{c}}\Big|_{t'=t_{ret}} \tag{1}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\frac{q\mathbf{V}(t')}{R(t') - \frac{\mathbf{V}(t')\cdot\mathbf{R}(t')}{c}}\Big|_{t'=t_{ret}}, \tag{2}$$

where t_{ret} is the solution of the equation $f(t') = t - t' - \frac{|\mathbf{x}-\mathbf{x}'|}{c} = 0$ and $\mathbf{R}(t) := \mathbf{x} - \mathbf{X}(t)$. Why does $f(t') = 0$ only have one solution? (Hint: $|\mathbf{V}(t)| < c$.) (4 points)

2. Calculate the Liénard-Wiechert potentials according to (1) and (2) for a point charge q moving at a constant velocity $\mathbf{v}_0 = v\mathbf{e}_z$ on the z-axis. Show that they are given by

$$\phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{\tilde{R}} \quad \text{and} \quad \mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\frac{qv}{\tilde{R}}\mathbf{e}_z,$$

with $\tilde{R} = \sqrt{(1-\beta^2)(x^2+y^2)+(z-vt)^2}$ and $\beta = \frac{v}{c}$. At time $t=0$ the charge is in the origin. (3 points)

3. Calculate the fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ for the case in part 2. (2 points)

Problem 2 (2 points): *boundary conditions*

Starting from the stationary Maxwell's equations, derive the behavior of the fields $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{B}$, $\mathbf{H}$ at the boundary between two media of dielectric constants ϵ_1 and ϵ_2 and permeabilities μ_1 and μ_2, respectively. Consider the tangential as well as the normal components. Restrict yourselves to the cases where the simple relations $\mathbf{D} = \epsilon\,\epsilon_0\mathbf{E}$ and $\mathbf{B} = \mu\mu_0\mathbf{H}$ are valid. (Hint: use the theorems of Gauß and Stokes!)

Problem 3 (9 points): *cavity in dielectric*

Consider a spherical cavity with radius R and center at the origin, which is surrounded by a dielectric with dielectric constant ϵ. Let the potential ϕ far away from the cavity ($r\to\infty$) be $\phi_\infty = \phi_1 + \phi_2$ with $\phi_1 = -E_0 z$ and $\phi_2 = -\frac{1}{2}F_0(3z^2 - r^2)$, E_0 and F_0 constants. Calculate the potential for this configuration in the following way:

1. Since there are no free charges, the potential obeys the Laplace equation $\Delta\phi = 0$. Make an ansatz for the potential inside (ϕ_i) and outside (ϕ_o) of the cavity, as done in the lecture (expansion in terms of Legendre polynomials). Analyze the behavior of the potentials at the origin and at infinity to restrict the coefficients. (2 points)

2. To determine the remaining coefficients, use the boundary conditions for the potentials at $r = R$:

$$\phi_i(R,\theta) = \phi_o(R,\theta) \qquad \partial_r\phi_i(r,\theta)|_{r=R} = \epsilon\partial_r\phi_o(r,\theta)|_{r=R}.$$

Use the fact that the Legendre polynomials are linearly independent. (3 points)

3. What are the potentials ϕ_i and ϕ_o? Show that the field which is induced in the dielectric by the cavity is a superposition of a dipole and quadrupole field. Write down the corresponding multipole moments $\mathbf{p}$ and $\mathbf{Q}$. Compare $\mathbf{E}_\infty$ and $\mathbf{E}_i$. (4 points)