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2nd Problem Set for Advanced Quantum Mechanics
winter term 2008

Problem 4 (Spin and density matrix) (2 Points)

The spin state of an electron is represented on $\mathbb{C}^2$ (in the basis formed by the eigenstates of $\hat{S}_z$) by the density matrix

$$\rho := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} ,$$

$a, b \in \mathbb{R}, a \geq 0, b \geq 0, a + b = 1$.

a) If a measurement is made of the Spin $\hat{S}_x$ what is the probability that the result will be
(i) $\hbar/2$
(ii) $-\hbar/2$?

b) Use these results to compute the expectation value of $\hat{S}_x$, and check that the answer agrees with the result calculated directly from the formula $\langle \hat{A} \rangle = \mathrm{tr}(\hat{\rho}\hat{A})$.

Problem 5 (Convexity of density matrices) (2 Points)

Let $\hat{\rho}_1$ and $\hat{\rho}_2$ be a pair of density matrices. Show that $r\hat{\rho}_1 + (1-r)\hat{\rho}_2$ is a density matrix for all $r \in \mathbb{R}$ such that $0 \leq r \leq 1$.

Problem 6 (Legendre polynomials) (4 Points)

The Legendre polynomials $P_l(x)$ can be defined by a generating function according to

$$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} P_l(x) t^l \,. \tag{1}$$

Prove from this equation by use of the Taylor formula the Rodrigues formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \,. \tag{2}$$

Show also that

$$\int_{-1}^{1} P_l(x) P_k(x) dx = \begin{cases} 0 & l \neq k \\ \frac{2}{2l+1} & l = k \end{cases} \tag{3}$$

(Hint: From (1) one gets

$$\frac{1}{\sqrt{1-2tx+t^2}} \cdot \frac{1}{\sqrt{1-2sx+s^2}} = \sum_{l,k=0}^{\infty} P_l(x) P_k(x) t^l s^k \, .$$

Integrate this from -1 to 1.)
Give the first four polynomials in explicit form.

Problem 7 (Eigensystem) (2 Points)

A particle of mass m moves in the potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ \frac{1}{2} m \omega^2 x^2 & \text{for } x > 0 \, . \end{cases}$$

Find the eigenvalues and eigenfunctions of the Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \, .$$

Deadline: Wednesday, 29.10.08