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3rd Problem Set for Advanced Quantum Mechanics

winter term 2008

Problem 8 (Spherical harmonics) (4 Points)

a) Expand the function

$$f(\vartheta,\varphi) = a + b\cos^2\vartheta + c\sin^2\vartheta + d\sin\vartheta\sin\phi \quad a,b,c,d \in \mathbb{C}$$

into spherical harmonics $Y_{lm}(\vartheta,\varphi)$.

b) The angular momentum operators $\vec{L}^2$ and L_z can be expressed in polar coordinates as

$$\begin{aligned} L_z &= -i\hbar\frac{\partial}{\partial\varphi} \\ \vec{L}^2 &= -\hbar^2\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)\right) \end{aligned}$$

Show by an explicit calculation that

$$\begin{aligned} L_zY_{lm} &= \hbar mY_{lm} \\ \vec{L}^2Y_{lm} &= \hbar^2 l(l+1)Y_{lm}\,. \end{aligned}$$

c) The parity operator is defined by

$$P\psi(\vec{x}) = \psi(-\vec{x})\,.$$

Show that $PY_{lm}(\vartheta,\varphi) = (-1)^l Y_{lm}(\vartheta,\varphi)$.

Problem 9 (Expansion of plane wave into spherical harmonics) (3 Points)

Prove the following expansion, which was used in the lecture:

$$e^{ikz} = e^{ikr\cos\vartheta} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\vartheta)$$

where $j_l(kr)$ are the spherical Bessel functions and $P_l(\cos\vartheta)$ are the Legendre polynomials. Hint: Use the orthogonality relations for the Legendre polynomials. Consider then the limit $kr \to \infty$ and the asymptotic formula for $j_l(kr)$ to find the coefficients $i^l(2l+1)$.

Problem 10 (Stationary perturbation theory) (4 Points)

A particle of mass m moves on a sphere with radius R. Show that the correct kinetic term reads

$$H_0 = \frac{\vec{L}^2}{2mR^2} \, ,$$

where $\vec{L}^2$ was defined in problem 8. In the presence of gravity one has to add the perturbation

$$H_1 = mgz = mgR\cos\vartheta \, .$$

Why are the exact eigenfunctions of the perturbed system characterised by the quantum number l and m_l? Calculate the first and second order corrections of the energy levels

$$E^{(1)}_{l,m_l} = \langle l, m_l | H_1 | l, m_l \rangle$$

and

$$E^{(2)}_{l,m_l} = \sum_{(l',m_{l'})\neq(l,m_l)} \frac{|\langle l', m_{l'} | H_1 | l, m_l \rangle|^2}{E^{(0)}_{l,m_l} - E^{(0)}_{l',m_{l'}}} \, .$$

Are the energy levels still degenerate?

Hint: $\cos\vartheta Y_{lm_l}(\vartheta,\varphi) = \sqrt{\frac{(l+1)^2 - m_l^2}{(2l+1)(2l+3)}} Y_{l+1m_l}(\vartheta,\varphi) + \sqrt{\frac{l^2 - m_l^2}{(2l+1)(2l-2)}} Y_{l-1m_l}(\vartheta,\varphi)$

Deadline:Wednesday, 5.11.08