Kaluza-Klein Theory

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Overview

References

- mainly J.M. Overduin and P.S.Wesson "Kaluza-Klein Gravity", arXiv:gr-qc/9805018v1 .
- T. Kaluza, "Zum Unitätsproblem der Physik", Sitz. Preuss. Akad. Wiss. Phys. Math. K1 (1921) 966.
- O. Klein, "Quantentheorie und fünfdimensionale Relativitätstheorie", Zeits. Phys. 37 (1926) 895.

Kaluza's idea

- in 1909 Lorentz generalized 3d-space and time to 4d-spacetime

  ⇒ general relativity could be constructed

- Generel Relativity explains gravity with pure geometry

- Kaluzas idea in 1921: use another dimension to construct a more general theory

Figure: Theodor Kaluza

Kaluza's idea

Three important aspects of Kaluza Theory:

1. Nature can be explained through pure geometry.
2. The underlying theory is a minimal extension of general relativity.
3. Physics only depends on the first four coordinates (called cylindricity).

   No mechanism is specified why it does not depend on the extra dimensions.

Notation

Notation in this presentation:

- capital latin indices A,B, $\ldots$ run over 0,1,2,3,4
- small greek indices $\alpha, \beta, \ldots$ run over 0,1,2,3
- hats denote 5d quantities ($\hat{G}$)

Basic ansatz

Ansatz: Einstein-equation in 5d:

$$\hat{G}_{AB} = 0$$

with

$$\begin{aligned}
\hat{G}_{AB} &= \hat{R}_{AB} - \frac{\hat{R}}{2}\hat{g}_{AB} \\
\hat{R}_{AB} &= \partial_C \hat{\Gamma}^C_{AB} - \partial_B \hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC} \\
\hat{\Gamma}^C_{AB} &= \frac{1}{2}\hat{g}^{CD}(\partial_A \hat{g}_{DB} + \partial_B \hat{g}_{DA} - \partial_D \hat{g}_{AB})
\end{aligned}$$

Basic ansatz and assumptions

- ansatz for the metric $\hat{g}_{AB}$:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\phi^2 A_\alpha A_\beta & \kappa\phi^2 A_\alpha \\ \kappa\phi^2 A_\beta & \phi^2 \end{pmatrix}$$

- physics does not depend on 5th dimension

  $\rightarrow$ all derivatives w.r.t. 5th dimension are 0

Resulting equations

One obtains:

$$G_{\alpha\beta} = \frac{\kappa^2\phi^2}{2} T_{\alpha\beta}^{\mathrm{EM}} - \frac{1}{\phi}[\nabla_\alpha(\partial_\beta\phi) - g_{\alpha\beta}\Box\phi]$$

$$\nabla^\alpha F_{\alpha\beta} = -3\frac{\partial^\alpha\phi}{\phi}F_{\alpha\beta}$$

$$\Box\phi = \frac{\kappa^2\phi^2}{4}F_{\alpha\beta}F^{\alpha\beta}$$

The scalar field

- if we assume $\phi = const.$ and with $\kappa = 4\sqrt{\pi G}$:

$$\begin{aligned} G_{\alpha\beta} &= 8\pi G\phi^2 T_{\alpha\beta}^{\mathrm{EM}} \\ \nabla^\alpha F_{\alpha\beta} &= 0 \end{aligned}$$

- originally Kaluza and Klein set $\phi = 1$

$\Rightarrow$ violates 3rd equation

Problem: extra dimensions

- have seen: can unify general relativity and electrodynamics through extra dimensions

  $\Rightarrow$ Problem: Why don't we see extra dimensions?

- solutions:

  1. compactification ($\rightarrow$ Kaluza-Klein Theory)
  2. projective geometry
  3. non-compactification

- every solution breaks one of the three important aspects of Kaluza Theory

Solutions for dimension

Kaluza-Klein theory: basic idea

- in 1926 Klein proposed compactification
- circular topology of 5th dimension

  ⇒ physical fields only depend periodically on 5th dimension
- small enough compactification scale

  ⇒ extra dimension is unobservable

Figure: Oskar Klein

Action

- modes carry momentum $\propto \frac{|n|}{r}$

  $\Rightarrow$ if r is small enough, only $n = 0$ contributes

- action:

$$S = -\frac{1}{16\pi \hat{G}} \int \hat{R} \sqrt{-\hat{g}} d^4 x dy$$

- periodicity of 5th dimension

  $\Rightarrow$ can expand 5d fields in Fourier modes

Metric

- parametrisation of the metric:

$$\hat{g}^{(0)}_{AB}(x) = \phi^{-1/3} \begin{pmatrix} g_{\alpha\beta} + \kappa^2 \phi A_\alpha A_\beta & \kappa\phi A_\alpha \\ \kappa\phi A_\beta & \phi \end{pmatrix}$$

- use Ricci tensor and Christoffel symbols as above and

$$\begin{aligned} \kappa &= 4\sqrt{\pi G} \\ G &= \hat{G} / \int dy \end{aligned}$$

Resulting action

Resulting action in 0th order Fourier expansion:

$$S = -\int d^4x\sqrt{-g}\left(\frac{R}{16\pi} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{6\kappa^2}\frac{\partial^\alpha\phi\partial_\alpha\phi}{\phi^2}\right)$$

Matter in Kaluza-Klein ansatz

- introduce scalar field in theory:

$$S_{\hat{\rho}} = -\int d^x dy \sqrt{-\hat{g}}\partial^A \hat{\rho} \partial_A \hat{\rho}$$

- expand $\hat{\rho}$ in Fourier-modes:

$$\hat{\rho}(x,y) = \sum_{n=-\infty}^{n=\infty} \hat{\rho}^{(n)}(x) e^{in\frac{y}{r}}$$

Resulting matter action

$$S_{\hat{\rho}} = -\left(\int dy\right)\sum_n \int d^4x\sqrt{-g}\left[\left(\partial^\alpha + \frac{in\kappa A^\alpha}{r}\right)\hat{\rho}^{(n)}\times \left(\partial_\alpha + \frac{in\kappa A_\alpha}{r}\right)\hat{\rho}^{(n)} - \frac{n^2}{\phi r^2}\hat{\rho}^{(n)2}\right]$$

- first term: covariant derivative
- second term: mass term

Charge and mass of matter fields

- comparison with covariant derivative from QED yields:

$$q_n = \frac{n\kappa}{r}\left(\phi\int dy\right)^{-1/2} = \frac{n\sqrt{16\pi G}}{r\sqrt{\phi}}$$

- mass term gives:

$$m_n = \frac{|n|}{r\sqrt{\phi}}$$

Achievements and problems

Special achievements:

- prediction of quantization of charge

Special problems:

- explicit addition of matter
- quantized mass
- predicted mass of lowest Fouriermode $\sim m_{\rm pl}$

  $\Rightarrow$ higher dimensions

Projective geometry

Projective geometry

Projective geometry: Basic Idea

- replaces classical tensors in general relativity with projective tensors
- extra dimensions are artifacts of underlying theory
- extra dimensions are regarded as visual aids

  $\Rightarrow$ extra dimensions are not physically real and thus cannot be observed
- sacrifices geometric interpretation of general relativity and is more than a minimal extension of general relativity

Problems in early attempts

- Kaluza theory gives rise to Brans-Dicke scalar field
- Brans-Dicke action:

$$S = -\int d^4x\sqrt{-g}\left(\frac{R\phi}{16\pi G} + \omega\frac{\partial^\alpha\phi\partial_\alpha\phi}{\phi}\right) + S_{\mathrm{m}}$$

with ω the Brans-Dicke constant and S_{m} the action of associated matter fields

Problems in early attempts

- Kaluza action without electromagnetic potential:

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R\phi$$

$\Rightarrow \omega = 0$

- ω is constrained by experiment to be greater than ~ 500

Solution attempt 1

- Lessner suggested that Brans-Dicke scalar is microscopic
- mathematically the same equations as in Kaluza theory

  except G is replaced by essentially free parameter B
- solution of field equations is called particle if it satisfies certain conditions

Solution attempt 2

- projective unified field theory by Schmutzer
- explicitly introduce matter substrate
- 5d Einstein equation:

$$\hat{G}_{AB} = 8\pi \hat{G} \hat{\theta}_{AB}$$

  with energy projector $\hat{\theta}_{AB}$

- use conformally rescaled metric $g'_{\alpha\beta} = e^{-\sigma} g_{\alpha\beta}$

resulting field equations

$$
\begin{aligned}
G_{\alpha\beta} &= 8\pi G(T^{\mathrm{EM}}_{\alpha\beta} + \Sigma_{\alpha\beta} + \theta_{\alpha\beta}) \\
\nabla_\alpha H^{\alpha\beta} &= J^\beta \\
\Box\sigma &= 8\pi G(\frac{2}{3}\vartheta + \frac{1}{2}F_{\alpha\beta}H^{\alpha\beta})
\end{aligned}
$$

with electric energy-momentum tensor $T^{\mathrm{EM}}_{\alpha\beta}$, substrate energy tensor $\theta_{\alpha\beta} = \hat{\theta}_{\alpha\beta}$ and ...

Some tensors

scalaric energy tensor:

$$\Sigma_{\alpha\beta} = -\frac{3}{16\pi G}\left(\partial_\alpha\sigma\partial_\beta\sigma - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\sigma\partial_\gamma\sigma\right)$$

and electric four-current J^α, electromagnetic field-strength tensor $F_{\alpha\beta}$, induction tensor $H_{\alpha\beta} = e^{3\sigma}F_{\alpha\beta}$ and the scalaric substrate density

$$\vartheta = e^{-\sigma}\hat{\theta}_A^A - \frac{3}{2}\theta_\alpha^\alpha$$

Implications

- violations of weak equivalence principle for time dependent scalar fields
- scalaric polarization of vacuum
- dependent on scalarism parameter $\gamma = \frac{\vartheta}{\rho}$ (main free parameter)
- experimental constraints are upper limits on γ

Achievements and problems

Special achievements:

- in principle testable predicitions

Special problems:

- modifies geometric foundation
- either needs substrate or can only describe microscopic phenomenons

Noncompactified theories

Noncompactified theories

Noncompactified theories: basic idea

- take extra dimension to be physical
- relax cylindricity condition
- perhaps not lengthlike extradimension

  $\Rightarrow$ effects are very small if unit conversion factor is small

Metric

- choose metric such that A_α vanishes

  possible because we are not imposing cylindricity

- include factor ϵ to allow time- and spacelike signature

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \epsilon\phi^2 \end{pmatrix}$$

First resulting field equation

- using $\hat{R}_{AB} = 0$ one gets:

$$R_{\alpha\beta} = \frac{\nabla_\beta(\partial_\alpha \phi)}{\phi} - \frac{\epsilon}{2\phi^2}\left(\frac{\partial_4\phi\partial_4 g_{\alpha\beta}}{\phi} - \partial_4(\partial_4 g_{\alpha\beta}) + g^{\gamma\delta}\partial_4 g_{\alpha\gamma}\partial_4 g_{\beta\delta} - \frac{g^{\gamma\delta}\partial_4 g_{\gamma\delta}\partial_4 g_{\alpha\beta}}{2}\right)$$

- can build Einstein-equation from equation for $R_{\alpha\beta}$
- contracting with $g^{\alpha\beta}$ gives R

First resulting field equation

- insert $R_{\alpha\beta}$ and R in

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$$

- results in energy-momentum tensor of induced matter:

$$\begin{aligned} 8\pi G T_{\alpha\beta} \quad = \quad & \frac{\nabla_\beta(\partial_\alpha \phi)}{\phi} - \frac{\epsilon}{2\phi^2} \left[\frac{\partial_4 \phi \partial_4 g_{\alpha\beta}}{\phi} - \partial_4(\partial_4 g_{\alpha\beta}) \right. \\ & + g^{\gamma\delta} \partial_4 g_{\alpha\gamma} \partial_4 g_{\beta\delta} - \frac{g^{\gamma\delta} \partial_4 g_{\gamma\delta} \partial_4 g_{\alpha\beta}}{2} \\ & \left. + \frac{g_{\alpha\beta}}{4} \left(\partial_4 g^{\gamma\delta} \partial_4 g_{\gamma\delta} + (g^{\gamma\delta} \partial_4 g_{\gamma\delta})^2 \right) \right] \end{aligned}$$

Second resulting field equation

- one gets:

$$\nabla_\beta P^\beta_\alpha = 0$$

with

$$P^\beta_\alpha = \frac{1}{2\sqrt{\hat{g_{44}}}} \left(g^{\beta\gamma} \partial_4 g_{\gamma\alpha} - \delta^\beta_\alpha g^{\gamma\epsilon} \partial_4 g_{\gamma\epsilon} \right)$$

- interpretation not clear
- could be related to known conserved quantity
- another possible identification allows to relate this equation with 4d geodesic equation

Third resulting field equation

- one gets:

$$\epsilon\phi\Box\phi = -\frac{\partial_4 g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{4} - \frac{g^{\alpha\beta}\partial_4(\partial_4 g_{\alpha\beta})}{2} + \frac{\partial_4\phi g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{2\phi}$$

- interpretation not clear
- can identify this equation with Klein-Gordon equation with appropriate definition of mass

Interpretation of 5th coordinate

- 5th coordinate might be related to rest mass
- hints for that:
  - basic mechanics depends on basis units of length, time and mass length and time are already merged
  - calculations with explicit 5d metrics show that metrics with dependency on 5th coordinate can be related to massive physical situations
  - realistic 4d metrics emerge from constant 5th coordinate hypersurfaces $\Rightarrow$ coordinate frame where rest mass is constant

Achievements and problems

Special Achievements:

- gives Energy-momentum tensor from geometry
- testable predictions

Special problems:

- physical quantities dependent on 5d coordinate-system
- interpretation not clear

Overview

Predictions

- 3 theories similar predictions

  $\Rightarrow$ not distinguishable

- for matter interpretation: slow variation of rest mass ($\sim 10^{-12}\,\mathrm{yr}^{-1}$)

- soliton solutions show deviations in classical tests, for example:
  - gravitational redshift
  - light deflection
  - geodetic precession $\rightarrow$ Gravity Probe-B is close

Conclusion

- have seen that general relativity and electrodynamics can be unified

  $\Rightarrow$ possibly unify other forces

- three different ways for extra dimensions:

  1. compactification (used e.g. in string theory)
  2. projective geometry
  3. non-compactification

- 5th dimension might be related to restmass

- none of the three approaches can be excluded on observational grounds at present time