# The Weyl Tensor

## Introduction, properties and applications

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Dec 06, 2016

# Overview

1. Motivation
2. Derivation and properties of the Weyl Tensor
3. Where the Weyl Tensor appears
4. Application: Classification of Spacetimes using the Weyl Tensor (Petrov Classification)

# Motivation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$$

- Algebraic equations for the **traces** of the Riemann Tensor
- **Determine** 10 components of the Riemann Tensor
- No direct visibility of curvature *propagation*

**Traceless part of $R_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $C_{\alpha\beta\gamma\delta}$.**

# Decomposition of Curvature Tensor

**Definition**

*For $A_{\alpha\beta}$ symmetric, define:*

$$A^*_{\alpha\beta\gamma\delta} = A_{\alpha\gamma}g_{\beta\delta} + A_{\beta\delta}g_{\alpha\gamma} - A_{\alpha\delta}g_{\beta\gamma} - A_{\beta\gamma}g_{\alpha\delta}$$

$A^*_{\alpha\beta\gamma\delta}$ fulfills:

1. $A^*_{\alpha\beta\gamma\delta} = -A^*_{\alpha\beta\delta\gamma}$
2. $A^*_{\alpha\beta\gamma\delta} = A^*_{\gamma\delta\alpha\beta}$
3. $A^*_{\alpha\beta\gamma\delta} + A^*_{\alpha\gamma\delta\beta} + A^*_{\alpha\delta\beta\gamma} = 0$

**Possible Ansatz**

$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + AR^*_{\alpha\beta\gamma\delta} + BRg^*_{\alpha\beta\gamma\delta}, \quad A, B \in \mathbb{R}.$

# The Weyl Tensor

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{n-2}(R_{\alpha\gamma}g_{\beta\delta} + R_{\beta\delta}g_{\alpha\gamma} - R_{\alpha\delta}g_{\beta\gamma} - R_{\beta\gamma}g_{\alpha\delta}) + \frac{1}{(n-1)(n-2)}R(g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta})$$

See for example: H.-J. Schmidt, Gravitoelectromagnetism and other decompositions of the Riemann tensor (2004, arXiv:gr-qc/04070531v1)

Get a propagation equation:

$$\nabla^\alpha C_{\alpha\beta\gamma\delta} = \frac{n-3}{n-2}\kappa\left(\nabla_\gamma T_{\beta\delta} - \nabla_\delta T_{\beta\gamma} - \frac{1}{n-1}(\nabla_\gamma T g_{\beta\delta} - \nabla_\delta T g_{\beta\gamma})\right)$$

See for example: Matthias Blau, Lecture Notes on General Relativity (ttp://www.blau.itp.unibe.ch/Lecturenotes.html)

# Weyl Tensor Properties

1. Same algebraic symmetries as Riemann Tensor
2. Traceless: $g^{\alpha\gamma} C_{\alpha\beta\gamma\delta} = 0$
3. Conformally invariant:
   - That means:

$$\tilde{g}_{\alpha\beta} = \Omega^2(x) g_{\alpha\beta} \quad \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} \quad \tilde{C}^{\alpha}{}_{\beta\gamma\delta} = C^{\alpha}{}_{\beta\gamma\delta}$$

   - $C = 0$ is sufficient condition for $g_{\alpha\beta} = \Omega^2 \eta_{\alpha\beta}$ in $n \geq 4$
4. Vanishes identically in $n < 4$
5. In vacuum it is equal to the Riemann tensor.

# Applications of the Weyl tensor

- Vacuum GR: Geodesic deviation, gravitational waves, ...
- Cosmology: Weyl Curvature Hypothesis and isotropic singularities [1]
  - Weyl Tensor is small at initial singularities [2]
  - $g = \Omega^2 g^*$, $g^*$ regular at $t = 0$:

$$\lim_{t \to 0} C^{\alpha}{}_{\beta\gamma\delta} < \infty, \quad \lim_{t \to 0} \frac{C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}}{R_{\mu\nu} R^{\mu\nu}} = 0$$

- Alternative theories of gravity:
  - $C^2$ gravity: $S = -\frac{\alpha}{4} \int \sqrt{-g} C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}$
  - Non-minimal coupling to other fields: [3] $S_{int} = \alpha \int \sqrt{-g} F_{\alpha\beta} F_{\gamma\delta} C^{\alpha\beta\gamma\delta}$

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[1] see e.g. Goode et. al 1992: http://iopscience.iop.org/article/10.1088/0264-9381/9/2/010/pdf

[2] Penrose 1979: General Relalivity: An Einstein Centenary Survey

[3] Chen et al.: Strong gravitational lensing for the photons coupled to Weyl tensor in a Kerr black hole spacetime, 2016

## The Petrov Classification

# The Petrov Classification: Mathematical Preliminaries I

## Definition (Bivector Space $\mathcal{B}(p)$)

*Space of second order skew symmetric contravariant tensors $X^{\alpha\beta}$ at point $p \in \mathcal{M}$:*

$$X \in \mathcal{B}(p) \rightarrow X^{\alpha\beta} = -X^{\beta\alpha}.$$

*$\mathcal{B}(p)$ is a 6-dimensional space.*

## Notation (Bivector indices)

$$\begin{aligned} X^{\alpha\beta}: \quad \alpha\beta &= \{01, 02, 03, 23, 31, 12\} \\ X^{A}: \quad A &= \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} \\ &\rightarrow \quad \epsilon_{AB} = \begin{pmatrix} 0 & \mathbb{I}_{3\times3} \\ \mathbb{I}_{3\times3} & 0 \end{pmatrix} \end{aligned}$$

# The Petrov Classification: Mathematical Preliminaries II

1. Weyl Tensor as linear map:

$$\begin{aligned} C : \mathcal{B}(p) &\rightarrow \mathcal{B}(p) \\ X^B &\mapsto C_{AB}X^B \end{aligned}$$

2. Eigenvalue equation:

$$C(X) = \lambda X, \quad \lambda \in \mathbb{C}$$

3. In coordinates

$$(C_{AB} - \lambda G_{AB})X^B = 0$$

$$G_{AB} \equiv G_{\alpha\beta\gamma\delta} = \eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma} = diag(-1, -1, -1, 1, 1, 1)$$

# The Petrov Classification: Mathematical Preliminaries III

## Definition (Hodge Duality Transformation)

$$\begin{aligned} {}^\star : \mathcal{B}(p) &\rightarrow \mathcal{B}(p) \\ X^A &\mapsto {}^\star X^A = \epsilon^A{}_B X^B \end{aligned}$$

## Definition (Hodge dual of Weyl Tensor)

*Weyl tensor as double-bivector → define left and right dual:*

$$\begin{aligned} {}^\star C_{AB} &:= \epsilon_A{}^D C_{DB} \\ C^\star_{AB} &:= \epsilon_B{}^D C_{AD} \end{aligned}$$

# The Petrov Classification: Magic of the Weyl Tensor I

**Can show:**

- Minkowski space: ${}^{\star\star}X = -X = X^{\star\star}$
- Duality property: $C({}^{\star}X) = C^{\star}(X)$
- Tracelessnes of the Weyl Tensor: ${}^{\star}C = C^{\star}$

## Definition ((Anti-)self-dual Bivector)

- $X \in \mathcal{B}(p)$ *is* **self-dual**, $X \in S^{+}(p)$, *if* ${}^{\star}X = -iX$
- $X \in \mathcal{B}(p)$ *is* **anti-self-dual**, $X \in S^{-}(p)$, *if* ${}^{\star}X = iX$

It follows for $X \in \mathcal{B}(p)$:

1. $X^{+} := X + i^{\star}X \in S^{+}(p)$
2. $X^{-} := X - i^{\star}X \in S^{-}(p)$
3. $X \in S^{+}(p) \Leftrightarrow X = \begin{pmatrix} A \\ iA \end{pmatrix}$

# The Petrov Classification: Magic of the Weyl Tensor II

The restricted Weyl tensor $C|_{S^+}$ is an automorphism on $S^+(p)$.
$C^+ = C + i^\star C$ maps $S^-(p)$ to zero:

$$\begin{aligned} X \in S^+(p) \to C^+(X) &= [C + i^\star C](X) = 2C(X) \\ X \in S^-(p) \to C^+(X) &= [C + i^\star C](X) = 0 \end{aligned}$$

For $C, X$ real, $C^-(X^-) = \overline{C^+(X^+)}$. **That is, it is enough to study the Eigenvalue equation of $C^+$, as they have to occur in complex conjugate pairs!**

# Petrov Classification: Constructing the Weyl Tensor

**Use:**

1. Symmetry $C_{AB} = C_{BA}$
2. Tracelessnes $C^{\alpha}{}_{\beta\alpha\delta} = 0$
3. Algebraic Bianchi Identity $C_{\alpha\beta\gamma\delta} + C_{\alpha\delta\beta\gamma} + C_{\alpha\gamma\delta\beta} = 0$

**One finds:**

$$C = \begin{pmatrix} M & N \\ N & -M \end{pmatrix} \rightarrow C^{+} = \begin{pmatrix} M - iN & N + iM \\ N + iM & -M + iN \end{pmatrix} := \begin{pmatrix} Q & -iQ \\ -iQ & -Q \end{pmatrix}$$

$M, N$ real traceless symmetric $3 \times 3$ matrices.
For $X \in S^{+}(p) \rightarrow X = (A, iA)^{T}$ for 3-dim vector $A$. We get the Eigenvalue equation $QA = \lambda A$.

# Finding eigenvalues

- Complex symmetric $3 \times 3$ matrix Q.
- Similarity transformations, $Q \to P^{-1}QP$. Same eigenvalues, rank, characteristic and minimal polynomials
- In general, Q not diagonalizable (only if real)
- Best we can do is construct the **Jordan normal form**
- Cayley-Hamilton Theorem: Every matrix fulfills its own characteristic equation

# Some linear algebra: Example 1

$$P^{-1}QP = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$$

$$\rightarrow \quad (a - \lambda)^2 (b - \lambda) = 0$$

Cayley-Hamilton $\rightarrow$ minimal polynomial:

$$(Q - a\mathbb{I})(Q - b\mathbb{I}) = 0$$

# Some linear algebra: Example 2

$$P^{-1}QP = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$$

$$\rightarrow \quad (a-\lambda)^2(b-\lambda) = 0$$

Cayley-Hamilton $\rightarrow$ minimal polynomial:

$$(Q - a\mathbb{I})^2(Q - b\mathbb{I}) = 0$$

Dimension of Eigenvectors of $\lambda = a$ (geometric multiplicity) is 1, less than the algebraic multiplicity 2!

# The Petrov Types

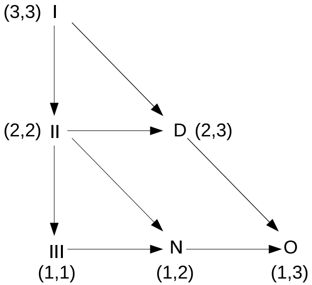
**Based on the Eigenvalues of the Matrix Q:**

From tracelessnes of the matrix Q:

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$

- Type I: All eigenvalues different $\rightarrow$ dimension of eigenspaces is 3
- Type D: One double eigenvalue with eigenspace-dimension 2
- Type II: One double eigenvalue with eigenspace-dimension 1
- Type III: All eigenvalues equal to zero. 1-dimensional eigenspace
- Type N: All eigenvalues equal to zero. 2-dimensional eigenspaces.
- Type O: All eigenvalues equal to zero. 3-dimensional eigenspaces, i.e. C=0

# Penrose Diagram for Petrov Types




(# different eigenvalues, # dimension of eigenvectors)

# The Petrov Types (other formulation)

**Based on the Eigenvalues of the Matrix Q:**

Use Caley-Hamilton theorem: Every similar matrix fulfills the same minimal polynomial.

| **Type** | **Eigenvalues** | **Minimal Polynomial** |
|---|---|---|
| I | $\{\lambda_1, \lambda_2, \lambda_3\}$ | $(Q - \lambda_1 I)(Q - \lambda_2 I)(Q - \lambda_3 I) = 0$ |
| D | $\{\lambda_1, \lambda_1, \lambda_2 = -2\lambda_1\}$ | $(Q - \lambda_1 I)(Q - \lambda_2 I) = 0$ |
| II | $\{\lambda_1, \lambda_1, \lambda_2 = -2\lambda_1\}$ | $(Q - \lambda_1 I)^2(Q - \lambda_2 I) = 0$ |
| III | $\{\lambda_1 = \lambda_2 = \lambda_3 = 0\}$ | $Q^3 = 0$ |
| N | $\{\lambda_1 = \lambda_2 = \lambda_3 = 0\}$ | $Q^2 = 0$ |
| O | $\{\lambda_1 = \lambda_2 = \lambda_3 = 0\}$ | $Q = 0$ |

# Some examples from physics

| Spacetime | Petrov Type |
|---|---|
| Spherically Symmetric spacetimes | D |
| Kerr | D |
| FLRW | O |
| Transversal gravitational waves | N |

# More physics...

Vacuum curvature tensor of an isolated matter distribution obeys under certain assumptions:

$$R_{\alpha\beta\gamma\delta} = \frac{N_{\alpha\beta\gamma\delta}}{r} + \frac{III_{\alpha\beta\gamma\delta}}{r^2} + \frac{D_{\alpha\beta\gamma\delta}}{r^3} + \dots$$

(See for example H. Stephani: Relativity, 2004)

# Literature


- G. S. Hall: Symmetries and Curvature Structure in General Relativity. World Scientific Publishing Company (April 29, 2004)
- Joan Mart inez i Portillo: Classification of Weyl and Ricci Tensors (Bachelor's Thesis, 2015/16) [4]
- A. Papapetrou: Lectures on general relativity. Springer Science & Business Media, 2012.
- H. Stephani: Relativity: An Introduction to Special and General Relativity, Cambridge University Press (March 29, 2004)


[4] https://upcommons.upc.edu/bitstream/handle/2117/87155/memoria.pdf?sequence=1&isAllowed=y

*That's it.*