University of Cologne www.thp.uni-koeln.de/gravitation/courses/rci2021.html
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Fifth exercise sheet on Relativity and Cosmology I

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Exercise 14 (6 points): *Curvature I*

For a nearly spherical body, the ratio of its Schwarzschild radius to its physical radius is a heuristic measure for the deviation of the geometry in the neighbourhood of the body from the flat Minkowski space-time.

14.1 Compare this ratio for a globular cluster of stars ($M \approx 10^6 M_\odot, R \approx 20\,\text{pc}$), the Sun, the Earth, a neutron star ($M \approx M_\odot$, $R \approx 10\,\text{km}$), a White Dwarf ($M \approx M_\odot$, $R \approx 10^4\,\text{km}$) as well as for a proton and an electron. For the latter two, use their Compton wavelengths $\hbar/mc$ as the (effective) radius.

14.2 Which mass would an elementary particle need to have, such that its Compton wavelength would be as large as its Schwarzschild radius? What size would its Schwarzschild radius then be?

14.3 The quantities appearing in these considerations are often expressed in terms of the so-called Planck units, which result from a unique combination of the natural constants G, c and $\hbar$. Calculate the Planck mass, the Planck length, the Planck time and the Planck energy in SI units.

Exercise 15 (9 points): *Curvature II*

In cylindrical coordinates (ρ, φ, z) of 3-dimensional Euclidean space, consider a surface of revolution with generatrix $z = \exp(-a^2\rho^2)$.

15.1 Determine the induced metric on the surface.

15.2 Calculate the curvature scalar at the apex using three different methods:

a) Compare the circumferences and the areas (Bertrand–Diguet–Puiseux theorem).

b) Find the radius of the spherical shell, that best approximates the given surface around the apex, and use the known curvature of a sphere with radius R.

Exercise 16 (6 points): *Transformations of the Christoffel symbols*

Consider the Christoffel symbols of the first and second kinds

$$\Gamma_{ikj} := \frac{1}{2}\Big(g_{ik,j} - g_{kj,i} + g_{ji,k}\Big), \qquad \Gamma^i{}_{kj} := g^{il}\Gamma_{lkj}.$$

and a general coordinate transformation $x^i \to x^{i'}(x^j)$

16.1 Derive the transformation of the symbols $\Gamma_{ikj} = \Gamma_{ikj}\Big(\Gamma_{i'k'j'}\Big)$ under the coordinate transformation. Do they constitute the components of a tensor?

16.2 Derive the corresponding transformation for $\Gamma^i{}_{kj} = \Gamma^i{}_{kj}\Big(\Gamma^{i'}{}_{k'j'}\Big)$.