University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Yi-Fan Wang, Leonardo Chataignier and Tim Schmitz

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Seventh exercise sheet on Relativity and Cosmology I

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Exercise 20 (6 points): *Riemannian normal coordinates*

Consider Riemannian normal coordinates y^μ at a point P, which is set as the coordinate origin $y^\mu = 0$.

22.1 Let $R_{\mu\kappa\lambda\nu}$ be the Riemann tensor. Show that

$$g_{\mu\nu}(y) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\kappa\lambda\nu}(0)\, y^\kappa\, y^\lambda + O\Big(|y|^3\Big)\,.$$

Show to this end first that

$$\Gamma^\mu{}_{\rho\sigma,\kappa}(0) + \Gamma^\mu{}_{\sigma\kappa,\rho}(0) + \Gamma^\mu{}_{\kappa\rho,\sigma}(0) = 0.$$

22.2 Give a physical interpretation of the equation.

Exercise 21 (4 points): *Algebraic identities of the Riemann tensor*

23.1 Let R_{ijkl} be the Riemann tensor. Show that the following algebraic identities hold

$$R_{ij(kl)} = 0\,, \quad R_{(ij)kl} = 0\,; \qquad R_{ijkl} + R_{iklj} + R_{iljk} = 0\,.$$

The last one is also called the *first* or *algebraic Bianchi identity*.

23.2 (bonus) Give interpretations of the first two identities in terms of parallel transport.

Exercise 22 (5 points): *Geometry and topology*

In Euclidean space $\mathbb{R}^3$, consider the cylindrical coordinates (ρ, φ, z). Define a surface of revolution $\mathcal{A}$ via

$$z^2 = [f(\rho)]^2\,, \qquad \rho^2 = x^2 + y^2\,,$$

where f is a strictly positive smooth function on $[0, a]$ with $f(a) = 0$, $f'(a^-) \to -\infty$ and $f'(0^+) = 0$.

24.1 Determine the induced metric $\mathrm{d}s^2 = h_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j$ on $\mathcal{A}$.

24.2 Let R be the Ricci scalar of $\mathcal{A}$. Evaluate the following integral explicitly

$$\int_{\mathcal{A}} \mathrm{d}^2 y\, \sqrt{h}\, R\,.$$

Exercise 23 (5 points): *Geodesic deviation*

Consider two neighbouring geodesics with worldlines $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu(\tau)$, where $\xi^\mu(\tau)$ is considered to be small, so that quadratic and higher-order terms can be neglected. Let $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ be the velocity. Show that the relative acceleration satisfies

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}\tau^2} = R^\mu{}_{\nu\kappa\lambda} u^\nu u^\kappa \xi^\lambda\,,$$

where $\frac{\mathrm{D}}{\mathrm{D}\tau} := u^\nu \nabla_\nu$, which is known as the *geodesic deviation equation*.