University of Cologne http://www.thp.uni-koeln.de/gravitation/courses/rci2223.html
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Tatevik Vardanyan, Mauricio Ortiz Torres and Blanca Hergueta ver. 1.02

Eighth exercise sheet on Relativity and Cosmology I

Winter term 2022/23

Release: Thu, Dec. 8th **Submit**: Thu, Dec. 15th **Discuss**: Thu, Jan. 12th

In the following exercises, consider a (pseudo-)Riemannian space with the Levi-Civita connection ∇.

Exercise 24 (6 points): *Killing equations*

24.1 Show that $2\nabla_{(\mu}\, v_{\nu)} = \pounds_{\underline{v}}\, g_{\mu\nu}$. What does $\pounds_{\underline{v}}\, g_{\mu\nu} = 0$ mean from a geometrical point of view?

24.2 Consider a Minkowski space in Cardesian coordinates, $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$. Find all Killing vector fields.

24.3 Consider a Poincaré half-plane $\mathrm{d}s^2 = (l/z)^2\big(\mathrm{d}x^2 + \mathrm{d}z^2\big)$, where l is a constant length, $z > 0$. Write down the Killing equations and (bonus) find all the Killing vector fields.

Exercise 25 (8 points): *Killing vector fields*

Let ξ^μ be a Killing vector field.

25.1 Let ξ^μ be time-like. Show that there exists a coordinate system (t, x, y, z), in which t is temporal, and the metric does not depend on time t, i.e. $\partial g_{\mu\nu}/\partial t = 0$ holds.

25.2 Let $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ be the tangent vector of an affine-parametrised geodesic. Show that $u^\mu \xi_\mu$ is constant along the geodesic. Interpret the physical meaning of the Killing vector fields from **24.2**, and compute their associated conserved quantities.

25.3 Let $T^{\mu\nu}$ be a symmetric tensor field with vanishing covariant divergence. Calculate $\big(\xi_\mu\, T^{\mu\nu}\big)_{;\nu}$.

25.4 (bonus) By using the first Bianchi identity and the Killing equations, prove

$$\xi_{\lambda;\kappa\nu} = -\,\xi_\mu\, R^\mu{}_{\nu\lambda\kappa}\,,$$

where $R^\mu{}_{\nu\lambda\kappa}$ is the Riemann tensor. Argue that the equation serves as an integrability condition.

Exercise 26 (6 points): *Fermi–Walker transport*

Let $x^i = x^i(s)$ be a curve, s its arc-length, and $u^i = \mathrm{d}x^i/\mathrm{d}s$ its tangent vector field. A vector v^i is called *Fermi–Walker transported* along the curve if

$$\frac{\mathrm{D}v^i}{\mathrm{D}s} + v_k\left(u^k\frac{\mathrm{D}u^i}{\mathrm{D}s} - \frac{\mathrm{D}u^k}{\mathrm{D}s}u^i\right) = 0\,.$$

26.1 If the curve is a geodesic, show that the Fermi–Walker transport is identical to the parallel transport.

26.2 Show that the tangent vector u^i is Fermi–Walker transported.

26.3 If v^i and w^i are Fermi–Walker transported, show that $v^i w_i$ is constant along the curve.

Remark. In practice, one may want to describe the motion of objects, which are more than simple point particles. The Fermi–Walker transport is convenient for describing non-rotating motions in the 3-dimensional sense. For instance, if spatial basis vectors were attached to a free gyroscope, they would be Fermi–Walker transported. This will later be employed in the discussion of *geodetic precession* and the so-called *Thirring–Lense effect*.