University of Cologne http://www.thp.uni-koeln.de/gravitation/courses/rci2223.html
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Tenth exercise sheet on Relativity and Cosmology I

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Exercise 30 (7 points): *Dust and ideal fluid*

In curved spacetime, the energy-momentum tensors of dust and ideal fluid are given by

$$T^{\mu\nu} = \rho\, u^\mu u^\nu + P\left(u^\mu u^\nu + g^{\mu\nu}\right),$$

where u^μ is the four-velocity field, ρ the energy density, P the pressure; for dust, $P = 0$.

30.1 Argue briefly that ρ and P are *scalars*.

30.2 For dust, show that dust particles move on geodesics.

30.3 Derive the continuity and the Euler equations of an ideal fluid by contracting $\nabla_\nu T^{\mu\nu} = 0$ with u_μ and $g_{\mu\nu} + u_\mu u_\nu$, respectively.

30.4 Consider the spatially-flat (*Friedmann–Lemaître–*)*Robertson–Walker metric*, defined by

$$\mathrm{d}s^2 = -N(t)^2\,\mathrm{d}t^2 + a(t)^2\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \qquad N > 0,\ a > 0.$$

Write down the continuity equation for an ideal fluid.

Hint: the spatial homogeneity and isotropy have to be used.

Exercise 31 (4 points): *Contracted Bianchi identity from action*

Let $R_{\mu\nu}$ and R be the Ricci tensor and scalar, respectively. Derive the contracted Bianchi identity

$$G^{\mu\nu}{}_{;\mu} := \left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0$$

by demanding the Einstein–Hilbert action be invariant under infinitesimal coordinate transformations.

Exercise 32 (3 points): *Relativistic charged particle I*

Consider a charged massive test particle in special relativity, described by the action ($c = 1$)

$$S\left[x^i\right] = \int_A^B \mathrm{d}t\, L\left(x^i, \dot{x}^j\right) := \int_A^B \mathrm{d}t\,\left\{-m\sqrt{1-\left(\dot{x}^i\right)^2} - q\Phi\left(x^j\right) + q\dot{x}^i A_i\left(x^k\right)\right\},$$

where $\dot{x}^i := \mathrm{d}x^i/\mathrm{d}t$, m and q are the mass and the electric charge, Φ and A_i the electric and vector potentials.

32.1 Calculate the *canonical* momentum $P_i = P_i\left(x^j, \dot{x}^k\right) := \partial L/\partial \dot{x}^i$. Derive its partial inverse $\dot{x}^i = v^i\left(x^j, P_k\right)$.

Remark. If such an inverse exists, the system is called *regular*, and there is *no constraint*.

32.2 (bonus) Calculate the *canonical* Hamiltonian $H = H\left(x^i, P_j\right)$. Derive the canonical equations of motion

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial P_i}\,, \qquad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}\,.$$

32.3 (bonus) From the results in **30.2**, find the relativistic Lorentz force in terms of the three-velocity $\dot{x}^i$, *kinematic* momentum $p_i := P_i - qA_i$, electric field $E_i := -\partial_i \Phi - \partial_t A_i$ and magnetic B-field $B^i := \epsilon^{ijk}\partial_j A_k$.

Exercise 33 (6 points): *Relativistic charged particle II: parametrised formulation*

Consider a charged massive test particle in special relativity, described by the action ($c = 1$)

$$S[x^\mu] = \int_A^B \mathrm{d}\lambda\, L(x^\mu, \dot{x}^\nu) := \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \left\{ -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + q\dot{x}^\mu A_\mu \right\}, \qquad \mu, \nu, \rho = 0, 1, 2, 3\,,$$

where $\dot{x}^\mu := \mathrm{d}x^\mu/\mathrm{d}\lambda$, A_μ is the four-potential.

33.1 Calculate the action under $\lambda \mapsto \lambda_f = f(\lambda)$, where the boundaries are fixed, $f(\lambda_{A,B}) = \lambda_{A,B}$, and $f'(\lambda) > 0$. Can one impose $\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$ before deriving the equations of motion?

Remark. Such a system is called *parametrised*, and belongs to a subset of all *singular systems*. The Einstein–Hilbert action is also parametrised.

33.2 Calculate the canonical four-momentum $P_\mu = P_\mu(x^\nu, \dot{x}^\rho) := \partial L/\partial \dot{x}^\mu$ of the particle. Show that its partial inverse $\dot{x}^\mu = v^\mu(x^\nu, P_\rho)$ does *not* exist.

Remark. Such a non-existence is the defining property of a *singular system*, which is often a synonym for *constrained system*. The Maxwell theory is also singular, but not parametrised.

33.3 Calculate $\dot{x}^\mu P_\mu(x^\nu, \dot{x}^\rho) - L(x^\nu, \dot{x}^\rho)$.

Remark. This result can be shown to be universal for all parametrised systems.