University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Nick Kwidzinski, Leonardo Chataignier and Yi-Fan Wang

www.thp.uni-koeln.de/gravitation/courses/rcii19.html

ver. 1.00

6^{th} exercise sheet on Relativity and Cosmology II

Summer term 2019

Release: Mon, May 13th **Submit**: Mon, May 20th in lecture **Discuss**: Thu, May 23rd/24th

Exercise 50 (12 credit points): *Reissner–Nordström solution*

The aim of this exercise is to calculate the the electromagnetic field and the spacetime metric outside of a static, spherically symmetric charge distribution with mass parameter M and charge parameter Q.

For this purpose, use the general spherically symmetric ansatz

$$\mathrm{d}s^2 = -\,\mathrm{e}^{\nu(t,r)}\,\mathrm{d}t^2 + \mathrm{e}^{\lambda(t,r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{1}$$

for the metric and calculate the electromagnetic field strength tensor $F^{\mu\nu}$, for example, by solving the inhomogeneous Maxwell equations ${F^{\nu\mu}}_{;\nu} = 4\pi j^\mu$. (*Check:* $F^{0r} = -F^{r0} = \mathrm{e}^{-(\lambda+\nu)/2}\,Q/r^2$, otherwise $F^{\mu\nu} = 0$.)

Afterwards, compute the components of the energy-momentum tensor given by

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\lambda}\,{F_\nu}^\lambda - \frac{1}{4}\,g_{\mu\nu}\,F_{\kappa\sigma}\,F^{\kappa\sigma}\right).$$

Now calculate the metric functions ν and λ by means of the Einstein equations ${G^\mu}_\nu = \varkappa\,{T^\mu}_\nu$.
(Use the components of the Einstein tensor ${G^\mu}_\nu$ arising from the ansatz in eq. (1) given in the lecture course.)

Choose the integration constant appropriately such that you obtain the correct Newtonian limit and compare the resulting metric to the Schwarzschild solution.

Exercise 51 (8 credit points): *Classical tests of GR for the Reissner–Nordström solution*

Find a coordinate transformation $\bar{r}(r)$ to transform the Reissner–Nordström metric into its isotropic form

$$\mathrm{d}s^2 = -\,A(\bar{r})\,\mathrm{d}t^2 + B(\bar{r})\left(\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\Omega^2\right),$$

where

$$A(\bar{r}) = \left[\left(1+\frac{M}{2\bar{r}}\right)^2 - \frac{Q^2}{4\bar{r}^2}\right]^{-2}\left[2+\frac{M}{\bar{r}} - \left(1+\frac{M}{2\bar{r}}\right)^2 + \frac{Q^2}{4\bar{r}^2}\right]^2 \quad \text{and} \quad B(\bar{r}) = \left[\left(1+\frac{M}{2\bar{r}}\right)^2 - \frac{Q^2}{4\bar{r}^2}\right]^2,$$

and calculate the coefficients α_1, α_2, β_1, β_2 introduced in the lecture, which are the coefficients of an expansion of $1/A(\bar{r})$ and $B(\bar{r})$ with respect to $u := 1/\bar{r}$:

$$\frac{1}{A(\bar{r})} = 1 + \alpha_1\,Mu + \alpha_2\,M^2u^2 + \mathcal{O}(u^3)\,, \quad B(\bar{r}) = 1 + \beta_1\,Mu + \beta_2\,M^2u^2 + \mathcal{O}(u^3)\,.$$

Use this result to analyse how a charge Q of the Sun would influence the classical tests of general relativity in the solar system.