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8th exercise sheet on Relativity and Cosmology II

Summer term 2019

Release: Mon, May 27th **Submit**: Mon, June 3rd in lecture **Discuss**: June 6th/7th

Exercise 54 (20 credit points): *Kerr–Newman metric*

The most general solution for a stationary black hole is given by the *Kerr–Newman metric*, which describes a black hole with angular momentum $J = M\,a$ and charge q. The line element expressed in Boyer–Lindquist coordinates takes the following form:

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}\left(\mathrm{d}t - a\sin^2(\theta)\,\mathrm{d}\phi\right)^2 + \frac{\sin^2(\theta)}{\rho^2}\left[\left(r^2+a^2\right)\mathrm{d}\phi - a\,\mathrm{d}t\right]^2 + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\,\mathrm{d}\theta^2\,,$$

where

$$\rho^2 = r^2 + a^2\cos^2(\theta)\,, \qquad \Delta = r^2 - 2Mr + q^2 + a^2\,, \qquad q^2 + a^2 \leqslant M^2\,.$$

54.1 How can one obtain this line element from the Kerr metric by means of a simple substitution of certain parameters (without calculation)?

54.2 For $\Delta = 0$ the metric exhibits coordinate singularities. Determine their radial coordinates $r_\pm$.

The surface $r_+ = \text{const.}$ (with r_+ being the radial coordinate with a larger value) represents the event horizon. Calculate its surface area for $t = \text{const.}$

54.3 Analogously to the Kerr metric, consider an observer with $r = \text{const.}$, $\theta = \pi/2$, whose tangent vector is parallel to the Killing field $\chi^\mu = \xi^\mu + \Omega\,\Psi^\mu$.

Which values can Ω take for given $r \geqslant r_+$? Show that at the horizon only one value Ω_H is possible and determine this value.

54.4 Consider the Killing field $\chi^\mu = \xi^\mu + \Omega\,\Psi^\mu$ evaluated at the event horizon.

Show that this Killing field is light-like on the entire horizon. Furthermore, show that the surface gravity κ defined by means of $\left[\nabla^\mu(\chi_\nu\chi^\nu)\right]_H = -2\,\kappa\,\chi^\mu\big|_H$ is a well-defined quantity.

Calculate the Lie derivative of the defining equation for κ with respect to χ^μ and thereby show that κ is constant along the integral curves of χ.

Remark: After a rather long calculation one obtains $\kappa = (r_+ - M)/(r_+^2 + a^2)$. (Not to be shown here.)

54.5 Consider the null geodesics defined at the horizon, whose tangent vectors k^μ are proportional to χ^μ. Find the functional relationship between the affine parameter λ of these null geodesics and the Killing parameter v of the integral curves of χ^μ, i.e. $\chi^\mu = (\partial/\partial v)^\mu$.