University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Nick Kwidzinski, Leonardo Chataignier and Yi-Fan Wang

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10th exercise sheet on Relativity and Cosmology II

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Release: Mon, June 17th **Submit**: Mon, June 24th in lecture **Discuss**: June 27th/28th

Exercise 58 (20 credit points): *Derivation of the Friedmann equations in Cartan calculus*

The aim of this exercise is to derive the Friedmann equations using the Cartan formalism.

We start with the Robertson–Walker line element in coordinates that is given by:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right]. \tag{1}$$

Remember that in terms of the pseudo-orthogonal coframe basis $\{\vartheta^i\}$, $i = 0, \dots, 3$, the metric takes the form

$$\mathrm{d}s^2 = \eta_{ij}\,\vartheta^i \otimes \vartheta^j = -\,\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3\,. \tag{2}$$

Like in exercise 46, Latin letters are used for anholonomic frame indices, whereas Greek letters are used for holonomic coordinate indices.

58.1 Write out the components of a suitable coframe basis. For convenience, use the definition $w := \sqrt{1-kr^2}$.

58.2 Calculate the exterior derivatives $\mathrm{d}\vartheta^i$. Insert these into the first Cartan structure equation

$$\mathrm{d}\vartheta^i + \omega^i{}_j \wedge \vartheta^j = 0 \tag{3}$$

to determine the 1-form-valued components $\omega^i{}_j$ of the connection.

58.3 Calculate the curvature 2-forms $\Omega^i{}_j$ by using the second Cartan structure equation

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_a \wedge \omega^a{}_j =: \frac{1}{2}\,R^i{}_{jkl}\,\vartheta^k \wedge \vartheta^l \tag{4}$$

and read off the anholonomic components $R^i{}_{jkl}$ of the Riemann curvature tensor.

Intermediate result: The non-vanishing anholonomic components of the Riemann curvature tensor read

$$R^r{}_{ttr} = -\,R^r{}_{trt} = R^\theta{}_{tt\theta} = -\,R^\theta{}_{t\theta t} = R^\phi{}_{tt\phi} = -\,R^\phi{}_{t\phi t} = \frac{\ddot{a}}{a}\,, \tag{5}$$

$$R^\theta{}_{r\theta r} = -\,R^\theta{}_{rr\theta} = R^\phi{}_{r\phi r} = -\,R^\phi{}_{rr\phi} = R^\phi{}_{\theta\phi\theta} = -\,R^\phi{}_{\theta\theta\phi} = \frac{\dot{a}^2 + k}{a^2}\,. \tag{6}$$

58.4 Determine the anholonomic components of the Ricci tensor $R_{ij} = R^a{}_{iaj}$ as well as the Ricci scalar $R = \eta^{ij}R_{ij}$. Note that for the contraction of anholonomic indices the Minkowski metric has to be used.

58.5 Calculate the mixed components of the Einstein tensor in the holonomic coordinate basis

$$G^i{}_j \stackrel{*}{=} G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\,\delta^\mu_\nu\,R\,. \tag{7}$$

58.6 Use the energy–momentum tensor of an ideal fluid with energy density ρ and pressure p given by

$$\{T^\mu{}_\nu\} = \mathrm{diag}(-\rho(t), p(t), p(t), p(t)) \tag{8}$$

to write out the Einstein equations $G^\mu{}_\nu = 8\pi G\,T^\mu{}_\nu$, which are called Friedmann equations in this case.