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12th exercise sheet on Relativity and Cosmology II

Summer term 2019

Release: Mon, July 1st **Submit**: Mon, July 8th in lecture **Discuss**: July 11th/12th

Exercise 62 (10 credit points): *Friedmann III*

Current observations indicate that we live in a flat ($\mathcal{K} = 0$) universe with positive cosmological constant, in which the contribution of radiation to the total energy density can be neglected, i. e. the only contributions come from non-relativistic matter (dust) and the cosmological constant.

Solve the Friedmann equation for this model. (*Hint:* The substitution $x^2 = (1/\Omega_{\mathrm{m},0} - 1)\, a^3$ could be helpful.) Determine the age of the universe as a function of H_0 and $\Omega_{\mathrm{m},0}$. How does $a(t)$ behave for large and small values of t?

According to well-established models for stellar evolution, several globular clusters in our galaxy are at least 12 billion years old. Draw a $(h - \Omega_\mathrm{m})$-diagram (h is the parameter in the definition of H_0) and sketch the contour lines for a constant age of the universe. Determine which parameter range is compatible with the above-mentioned observation. In doing so, only consider values $0.4 < h < 1$.

Current observations by the Planck satellite indicate that in the present universe $\Omega_{\mathrm{m},0} \approx 0.31$ and $\Omega_\mathrm{v} \approx 0.69$.

Calculate the redshift at which the energy density of matter was equal to that of the vacuum. Compare this to the redshift at which $\ddot{a}$ was equal to zero.

Exercise 63 (10 credit points): *Dark energy*

One way to simulate a cosmological constant is by means of a homogeneous scalar field ϕ with a suitable potential $V(\phi)$. For this purpose, consider the action

$$S = \int \mathrm{d}^4x\, \sqrt{-g} \left(\frac{1}{2}\, g^{\mu\nu}\, \partial_\mu \phi\, \partial_\nu \phi - V(\phi) \right).$$

63.1 Derive the equation of motion for a homogeneous field $\phi(t)$ in a Friedmann universe.

63.2 Calculate the energy–momentum tensor of the scalar field by means of a variation with respect to the metric. Specialize this calculation to a homogeneous field in a Friedmann universe and identify its energy–momentum tensor with that of an ideal fluid. That way, determine the energy density ρ_ϕ and the pressure p_ϕ. For which idealization does ϕ describe a cosmological constant?

63.3 In a concrete model one considers the potential $V(\phi) = \kappa/\phi^\alpha$ with at first arbitrary parameters κ and α. The scale factor shall obey the time evolution $a(t) \propto t^n$ (universe with $\mathcal{K} = 0$; $n = \frac{2}{3}$ during matter domination, $n = \frac{1}{2}$ during radiation domination).

Look for a solution for ϕ of the form $\phi(t) \propto t^A$. Determine A and find the relation that has to be imposed between κ and α. Finally, calculate the energy density ρ_ϕ and compare this to the density ρ of matter (or radiation, respectively).