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Sixth exercise sheet on Relativity and Cosmology II

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Exercise 46 (6 credit points): *Kruskal–Szekeres coordinates*

46.1 Derive the line element of the Schwarzschild metric in Kruskal–Szekeres coordinates as given in the lecture

$$\mathrm{d}s^2 = \frac{32(GM)^3}{r}\mathrm{e}^{-r/2GM}(-\mathrm{d}T^2 + \mathrm{d}X^2) + r^2\,\mathrm{d}\Omega^2. \tag{1}$$

For this purpose, introduce a *tortoise* radial coordinate (for $r > 2GM$) as follows

$$r_* = r + 2GM\ln\left(\frac{r}{2GM} - 1\right). \tag{2}$$

Then perform the coordinate transformation:

$$X = \exp\left(\frac{r_*}{4GM}\right)\cosh\left(\frac{t}{4GM}\right), \quad T = \exp\left(\frac{r_*}{4GM}\right)\sinh\left(\frac{t}{4GM}\right). \tag{3}$$

46.2 In the previous semester you have seen a similar transformation to a line element which had nothing to do with the Schwarzschild solution. Find this line element and the corresponding diagram, and compare it with the Schwarzschild metric in Kruskal–Szekeres coordinates in eq. (2).

Exercise 47 (5 credit points): *Another set of coordinates for the Schwarzschild spacetime*

Construct a coordinate system for the Schwarzschild metric that is singularity-free at the event horizon by transforming the Schwarzschild time t according to

$$t \to T = t + f(r). \tag{4}$$

Determine $f(r)$ by imposing that the prefactor of dr^2 is equal to $+1$ in the transformed line element. Write out the transformed line element. Is it still static? Which parts of the Kruskal diagram are covered by these coordinates?

Exercise 48 (9 credit points): *Penrose diagrams*

48.1 Express the line element for Minkowski spacetime in terms of spherical coordinates (t, r, θ, ϕ). Then perform a coordinate transformation

$$u = t - r, \quad v = t + r. \tag{5}$$

Write out the transformed line element. How can one interpret the coordinates u and v?

48.2 Perform another coordinate transformation $(u, v) \mapsto (u', v')$ according to

$$u' = \arctan(u) =: t' - r', \quad v' = \arctan(v) =: t' + r'. \tag{6}$$

Draw a (t', r') diagram and hatch the area covered by these coordinates. Then draw a radial light ray in this diagram that goes from infinity (in the original coordinates) to $r = 0$ and back to infinity.
In a second (t', r') diagram, sketch the areas $t = \text{const.}$ and $r = \text{const.}$

48.3 Calculate the line element in the primed coordinates and show that it is conformal to the line element

$$d\bar{s}^2 = -4\left(dt'^2 - dr'^2\right) + \sin^2(2r')\, d\Omega^2 . \tag{7}$$