

Seventh exercise sheet on Relativity and Cosmology II

Summer term 2021

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Exercise 49 (15 credit points): *Interior solution for spherically symmetric stars*

On the fifth sheet, in exercise 45 you have started from the general line element of a spherically symmetric spacetime,

$$ds^2 = -e^{2a(r,t)} dt^2 + e^{2b(r,t)} dr^2 + r^2 d\Omega^2, \quad (1)$$

and solved the Einstein equations for vacuum, resulting in the Schwarzschild spacetime. In this exercise, consider instead of vacuum a stationary ideal fluid

$$T^\mu{}_\nu = -(\rho(r) + p(r)) \delta_0^\mu \delta_\nu^0 + p(r) \delta_\nu^\mu, \quad (2)$$

with mass density $\rho(r)$ and pressure $p(r)$.

49.1 Starting from the Einstein tensor for a general spherically symmetric metric derived for exercise 45, show that the Einstein equations reduce to

$$e^{2b(r)} = \left(1 - \frac{2GM(r)}{r}\right)^{-1}, \quad \text{with} \quad M(r) = M_0 + 4\pi \int_0^r d\tilde{r} \rho(\tilde{r}) \tilde{r}^2, \quad (3)$$

$$a(r) = -b(r) - 4\pi G \int_r^\infty d\tilde{r} e^{2b(\tilde{r})} \tilde{r} (\rho(\tilde{r}) + p(\tilde{r})). \quad (4)$$

You do not need to consider $G^2{}_2 = 8\pi G T^2{}_2$ and $G^3{}_3 = 8\pi G T^3{}_3$. They can be reduced to the Tolman–Oppenheimer–Volkoff equation, which will be derived below in a different way.

49.2 Show that one can recover the Schwarzschild solution by setting $\rho = p = 0$. To describe a star in equilibrium, what value should be chosen for the constant M_0 , and why?

49.3 Finally, derive the Tolman–Oppenheimer–Volkoff equation from the covariant conservation of the energy-momentum tensor,

$$\frac{dp}{dr} = -\frac{G(M(r) + 4\pi r^3 p(r))}{r^2 \left(1 - \frac{2GM(r)}{r}\right)} (\rho(r) + p(r)). \quad (5)$$

Is it possible to have a pressureless spherically symmetric star in equilibrium?

Exercise 50 (5 credit points): *Misner–Sharp mass*

A notion of mass useful for spherically symmetric stars is the Misner–Sharp mass. For spacetimes with line elements of the form

$$ds^2 = \gamma_{IJ} dx^I dx^J + r^2 d\Omega^2, \quad (6)$$

where I, J do not run over the angular variables (θ, ϕ) , it is defined as

$$M_{\text{MS}} = \frac{r}{2G} \left(1 - \gamma^{IJ} r_{,I} r_{,J}\right). \quad (7)$$

50.1 Evaluate the Misner–Sharp mass for line elements of the form (1). What values does it take for the solution from exercise 49 and for the Schwarzschild solution?

50.2 From the Einstein equations for (1) you computed for exercise **45**, show that under the assumption $T_{00} \geq 0$ the Misner–Sharp mass is everywhere non-decreasing with r .

Note that due to the possible occurrence of so-called trapped surfaces, this is not sufficient to prove that the Misner–Sharp mass is always positive. The full proof is much more involved, and also requires a stronger restriction on the energy-momentum tensor.