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Eleventh exercise sheet on Relativity and Cosmology II

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Exercise 56 (10 credit points): *Ideal fluids in Cosmology*

Consider an ideal fluid in a Friedmann–Lemaître model, with Robertson–Walker geometry governed by the Friedmann equations.

56.1 Show that comoving observers move on geodesics.

56.2 What is the energy-momentum tensor of the ideal fluid for this kind of observers? Under which conditions can it act like a cosmological constant in the Einstein equations?

56.3 Evaluate the covariant conservation of the energy-momentum tensor and show that this yields only *one* non-trivial equation, which on the other hand can also be deduced directly from the Friedmann equations.

56.4 Consider an equation of state of the form $p = w\rho$ with $w = \text{const.}$ Calculate the function $\rho(a)$. For which values of w does $\ddot{a} > 0$ hold? In which cases is $\rho + 3p > 0$ (the strong energy condition) fulfilled?

56.5 Calculate $\rho(a)$ for a so-called "Chaplygin gas" whose equation of state is $p = -A/\rho$ ($A = \text{const.} > 0$) and discuss the extremal cases $a \to 0$ and $a \to \infty$.

Exercise 57 (10 credit points): *Dark energy*

One way to simulate a cosmological constant is by means of a homogeneous scalar field ϕ with a suitable potential $V(\phi)$. For this purpose, consider the action

$$S = \int \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu\phi \, \partial_\nu\phi - V(\phi) \right).$$

57.1 Derive the equation of motion for a homogeneous field $\phi(t)$ in a Friedmann universe.

57.2 Calculate the energy-momentum tensor of the scalar field by means of a variation with respect to the metric. Specialize this calculation to a homogeneous field in a Friedmann universe and identify its energy-momentum tensor with that of an ideal fluid. That way, determine the energy density ρ_ϕ and the pressure p_ϕ. For which idealization does ϕ describe a cosmological constant?

57.3 In a concrete model one considers the potential $V(\phi) = \kappa/\phi^\alpha$ with arbitrary parameters κ and α. Consider the case where, aside from the scalar field, dust is present in a flat ($k = 0$) Friedmann universe, and the dust density is much higher than the contribution of the scalar field such that the scale factor behaves like $a(t) \propto t^{2/3}$. Show that under these assumptions that

$$w = \frac{p_\phi}{\rho_\phi} = -\frac{2}{2+\alpha}.$$

To this end, solve the equation of motion of the scalar field on the background of the dust dominated, flat Friedmann universe, making the ansatz $\phi(t) = B\, t^A$.