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Twelfth exercise sheet on Relativity and Cosmology II

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Exercise 58 (6 credit points): *Friedmann I*

Consider a Friedmann–Lemaître model with $\mathcal{K} \neq 0$ and present density parameters $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{r},0}$ and $\Omega_{\mathrm{v},0}$ as well as $\Omega := \Omega_{\mathrm{m},0} + \Omega_{\mathrm{r},0} + \Omega_{\mathrm{v},0}$. Furthermore, let

$$\rho_{\mathrm{c}}(a) = \frac{3\,\dot{a}^2}{8\pi G\,a^2}$$

be the critical density at a time when the scale parameter had the value a, and let $\Omega_{\mathrm{m}}(a) = \rho_{\mathrm{m}}(a)/\rho_{\mathrm{c}}(a)$ etc. be the corresponding relative densities.

Determine the quantity $\Omega(a) - 1$ as a function of $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{r},0}$, $\Omega_{\mathrm{v},0}$ and a. This quantity indicates how much the considered model "deviates" from a flat model at a certain time. What kind of problem with regard to the deviation from flatness at earlier times arises for a Friedmann–Lemaître model whose density parameter Ω differs only slightly from unity today?

Exercise 59 (7 credit points): *Friedmann II*

Solve the Friedmann equation for a universe that contains radiation as well as non-relativistic matter (dust). For this purpose, rewrite the Friedmann equation as a differential equation with respect to the conformal time η, solve this equation for the three possible values of $\mathcal{K}$ and write out the result in the form $(a(\eta), t(\eta))$.

Determine the t-dependence of $a(t)$ for early ($t \to 0$) and late ($t \to \infty$) times for all the possible values of $\mathcal{K}$.

Exercise 60 (7 credit points): *Friedmann III*

Current observations indicate that we live in a flat ($\mathcal{K} = 0$) universe with positive cosmological constant, in which the contribution of radiation to the total energy density can be neglected, i. e. the only contributions come from non-relativistic matter (dust) and the cosmological constant.

Solve the Friedmann equation for this model. (*Hint:* The substitution $x^2 = (1/\Omega_{\mathrm{m},0} - 1)\,a^3$ could be helpful.) Determine the age of the universe as a function of H_0 and $\Omega_{\mathrm{m},0}$. How does $a(t)$ behave for large and small values of t?

According to well-established models for stellar evolution, several globular clusters in our galaxy are at least 12 billion years old. Draw a $(h - \Omega_{\mathrm{m},0})$-diagram ($h$ is the parameter in the definition of H_0) and sketch the contour lines for a constant age of the universe. Determine which parameter range is compatible with the above-mentioned observation. In doing so, only consider values $0.4 < h < 1$.

Current observations by the Planck satellite indicate that in the present universe $\Omega_{\mathrm{m},0} \approx 0.31$ and $\Omega_{\mathrm{v}} \approx 0.69$.

Calculate the redshift at which the energy density of matter was equal to that of the vacuum. Compare this to the redshift at which $\ddot{a}$ was equal to zero.