University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Tatevik Vardanyan and Mauricio Ortiz Torres

http://www.thp.uni-koeln.de/gravitation/courses/rcii23.html

ver. 1.00

Fourth exercise sheet on Relativity and Cosmology II

Summer term 2023

Release: Thu, May 4th **Submit**: Thu, May 11th **Discuss**: Thu, May 25th

Exercise 43 (15 credit points): *Differential forms*

43.1 Consider a n-dimensional manifold with a metric. Let $\{\omega^i\}$ be an orthonormal co-basis of 1-forms, and let ω be the preferred volume form $\omega = \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^n$.
Show that, in an arbitrary coordinate system $\{x^k\}$, the following holds:

$$\omega = \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n\,, \tag{1}$$

where g denotes the determinant of the metric, whose components g_{ij} are given in these coordinates.

43.2 The contraction of a p-form ω (with components $\omega_{ij\ldots k}$) with a vector v (with components v^i) is given by $[\omega(v)]_{j\ldots k} = \omega_{ij\ldots k}\, v^i$. Consider the n-form $\omega = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$.

Show that, with a given vector field v, the following holds:

$$d[\omega(v)] = v^i{}_{,i}\, \omega\,. \tag{2}$$

43.3 Define $(\mathrm{div}_\omega v)\, \omega := d[\omega(v)]$.

Show that, by using coordinates in which ω has the form $\omega = f\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, the following holds:

$$\mathrm{div}_\omega v = \frac{1}{f}\left(f v^i\right)_{,i}\,. \tag{3}$$

43.4 In three-dimensional Euclidean space, the preferred volume form is given by $\omega = dx \wedge dy \wedge dz$.

Show that, in spherical coordinates, this volume form is given by $\omega = r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi$.
Use the result of **43.3** to show that the divergence of a vector field

$$v = v^r \frac{\partial}{\partial r} + v^\theta \frac{\partial}{\partial \theta} + v^\phi \frac{\partial}{\partial \phi}, \tag{4}$$

is given by

$$\mathrm{div}\, v = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 v^r\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta\, v^\theta\right) + \frac{\partial v^\phi}{\partial \phi}\,. \tag{5}$$

Exercise 44 (5 credit points + *8 bonus points*): *Electrodynamics in flat space-time*

Differential forms are a convenient tool for field theories, as we will show in this exercise on the example of Maxwell electrodynamics in Minkowski space-time. We know that electromagnetic field strength is given by Faraday antisymmetric tensor $F_{\mu\nu}$, and the current is given by j^μ. In the language of exterior calculus, $F_{\mu\nu}$ are components of the Faraday 2-form $\mathbf{F}$ describing an arbitrary electromagnetic field, given by

$$\mathbf{F} := \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = -E_x\, dt \wedge dx - E_y\, dt \wedge dy - E_z\, dt \wedge dz + B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy\,, \tag{6}$$

while j^μ are components of the current 1-form $\mathbf{j}$ is given by

$$\mathbf{j} := j_\mu\, dx^\mu = -\rho\, dt + j_x\, dx + j_y\, dy + j_z\, dz\,. \tag{7}$$

44.1 The Hodge star operator $\star$ maps p-forms to $(4-p)$-forms. Therefore, 2-forms are mapped to 2-forms by this operator. The 2-form dual to the Faraday 2-form is the Maxwell 2-form, defined by $\mathbf{G} := \star\mathbf{F}$.

The Hodge star operator acts on the 2-form basis as follows

$$\star(dx^\mu \wedge dx^\nu) = \frac{1}{2}\,\eta^{\mu\alpha}\eta^{\nu\beta}\varepsilon_{\alpha\beta\gamma\delta}dx^\gamma \wedge dx^\delta\,, \tag{8}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the inverse Minkowski metric and $\varepsilon_{\alpha\beta\gamma\delta}$ totally anti-symmetric Levi-Civita tensor density ($\varepsilon_{0123} = +1$). For example,

$$\star(dt \wedge dx) = -dy \wedge dz\,, \quad \text{etc.} \tag{9}$$

Show that the Maxwell 2-form is given by

$$\mathbf{G} = B_x\, dt \wedge dx + B_y\, dt \wedge dy + B_z\, dt \wedge dz + E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy\,. \tag{10}$$

Which relation between the components of $\mathbf{F}$ and $\mathbf{G}$ holds?

44.2 ***Bonus exercise.*** With these definitions, the Maxwell equations can be written in a compact form: $d\mathbf{F} = 0$ and $d\mathbf{G} = 4\pi \star \mathbf{j}$, therefore,

a) Show that the equation $d\mathbf{F} = 0$ corresponds to the two homogeneous Maxwell equations

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{E} + \partial_t\vec{B} = 0\,. \tag{11}$$

b) Calculate $\star\mathbf{j}$ (a 3-form), which is the dual of the current 1-form $\mathbf{j}$. For that, use the following relations

$$\star dx^\mu = \frac{1}{3!}\eta^{\mu\alpha}\varepsilon_{\alpha\beta\gamma\delta}\, dx^\beta \wedge dx^\gamma \wedge dx^\delta\,, \tag{12}$$

for example

$$\star dt = -dx \wedge dy \wedge dz\,, \quad \text{etc.} \tag{13}$$

c) Then show that the equation $d\mathbf{G} = 4\pi \star \mathbf{j}$ corresponds to the two inhomogeneous Maxwell equations

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \quad \text{and} \quad \vec{\nabla}\times\vec{B} - \partial_t\vec{E} = 4\pi\vec{j}\,. \tag{14}$$

d) What can you do to turn eq. (11) into eq. (14) in the vacuum case?

44.3 ***Bonus exercise.*** The exterior derivative is nilpotent, i.e. the relation $d(d\boldsymbol{\omega}) = 0$ holds for any p-form $\boldsymbol{\omega}$. Choosing $\boldsymbol{\omega} = \star\mathbf{j}$, show that this yields the continuity equation $\partial_t\rho - \vec{\nabla}\cdot\vec{j} = 0$.

44.4 ***Bonus exercise.*** Explain in your own words why a covariant derivative is needed on a curved background, and give special attention to the connection. What is the intuitive interpretation of the connection? Why can it be chosen to be zero in Minkowski space-time?