

## Fifth exercise sheet on Relativity and Cosmology II

Summer term 2023

**Release:** Thu, May 11<sup>th</sup>

**Submit:** Thu, May 25<sup>th</sup>

**Discuss:** Wed, Jun. 7<sup>th</sup>

### Exercise 45 (20 credit points): *Derivation of the Schwarzschild solution in Cartan calculus*

The aim of this exercise is to derive the Schwarzschild solution using the Cartan formalism.

The general spherically symmetric ansatz is given by:

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = -e^{2a(r,t)} dt^2 + e^{2b(r,t)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (1)$$

In terms of the pseudo-orthogonal coframe basis  $\{\vartheta^i\}$ ,  $i = 0, \dots, 3$ , the metric takes the form

$$ds^2 = \eta_{ij} \vartheta^i \otimes \vartheta^j = -\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3. \quad (2)$$

In this exercise, numerals (0, 1, 2, 3) and Latin indices ( $i, j, k, l, \dots$ ) indicate an anholonomic basis, whereas coordinate ( $t, r, \theta, \phi$ ) and Greek indices ( $\mu, \nu, \lambda, \sigma, \dots$ ) refer to a holonomic basis. Holonomic and anholonomic indices can be converted into each other by using the *tetrad* defined via

$$\vartheta^i = e_\mu^i dx^\mu. \quad (3)$$

**45.1** Show that

$$\vartheta^0 = e^{a(r,t)} dt, \quad \vartheta^1 = e^{b(r,t)} dr, \quad \vartheta^2 = r d\theta, \quad \vartheta^3 = r \sin \theta d\phi \quad (4)$$

defines a suitable orthonormal coframe basis to describe the spherically symmetric ansatz.

**45.2** Now, calculate the exterior derivatives  $d\vartheta^i$ . Insert these into the first Cartan structure equation

$$\Theta^i := d\vartheta^i + \omega^i_j \wedge \vartheta^j = 0 \quad (5)$$

and show that the connection  $\omega^i_j$  in components reads

$$\begin{aligned} \omega^1_0 &= a' e^{-b(t,r)} \vartheta^0 + \dot{b} e^{-a(t,r)} \vartheta^1, & \omega^3_1 &= \frac{e^{-b(t,r)}}{r} \vartheta^3, & \omega^3_2 &= \frac{\cot \theta}{r} \vartheta^3, \\ \omega^2_1 &= \frac{e^{-b(t,r)}}{r} \vartheta^2, & \omega^3_0 &= 0, & \omega^2_0 &= 0, \end{aligned}$$

where a prime and superscript dot denote derivatives with respect to  $r$  and  $t$ , respectively.

*Hint:* For any metric-compatible connection (like the Levi-Civita connection studied here) we have  $\nabla g_{ij} = 0$  and therefore  $\omega_{ij} = -\omega_{ji}$ .

**45.3** Calculate the curvature 2-forms  $\Omega^i_j := d\omega^i_j + \omega^i_a \wedge \omega^a_j$ . Use the second Cartan structure equation

$$\Omega^i_j = \frac{1}{2} R^i_{jkl} \vartheta^k \wedge \vartheta^l \quad (6)$$

to show that the non-vanishing anholonomic components  $R^i_{jkl}$  of the Riemann curvature tensor are

$$\begin{aligned} -R^0_{101} &= e^{-2b}(a'^2 - a'b' + a'') + e^{-2a}(\dot{a}\dot{b} - \dot{b}^2 - \ddot{b}) = R^1_{010}, \\ R^0_{202} = R^0_{303} &= -\frac{a' e^{-2b}}{r}, \quad R^1_{212} = R^1_{313} = \frac{b' e^{-2b}}{r}, \\ R^0_{212} = R^0_{313} &= -\frac{\dot{b} e^{-a-b}}{r} = -R^1_{202} = -R^1_{303}, \quad R^3_{232} = \frac{1 - e^{-2b}}{r^2}. \end{aligned}$$

**45.4** Determine the anholonomic components of the Ricci tensor  $R_{ij} = R^k_{ikj}$  as well as the Ricci scalar  $R = \eta^{ij} R_{ij}$ . Note that for the contraction of anholonomic indices the Minkowski metric has to be used.

**45.5** Calculate the *holonomic* components of the Einstein tensor  $G^\mu_\nu = R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R$ . In general, one can find the holonomic components of a tensor via the tetrad,  $A_{\mu\nu} \stackrel{*}{=} e_\mu^i e_\nu^j A_{ij}$ . However, for a diagonal metric, the holonomic diagonal components of a (1,1)-tensor coincide with the anholonomic diagonal components, so one can directly identify  $A^i_i \stackrel{*}{=} A^\mu_\mu$ , not summing over the respective indices.

**45.6** Back to physics. Outside of the mass distribution we have a vacuum and therefore  $T^\mu_\nu = 0$ , i.e.  $G^\mu_\nu = 0$ . Use this to show that  $b$  depends only on  $r$  and that  $a(r) = -b(r)$ .

**45.7** Finally, integrate the differential equation arising from  $G^t_t = 0$  to find a relation between  $b$  and  $r$  and use the Newtonian limit to fix the integration constant. Show that

$$e^{-2b(r)} = 1 - \frac{2GM}{r}, \quad (7)$$

which concludes our derivation of the Schwarzschild solution.