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www.thp.uni-koeln.de/gravitation/courses/rci12.html

10th exercise sheet on Relativity and Cosmology I

Winter term 2012/13

Deadline for delivery: Thursday, 20th December 2012 during the exercise class.

Exercise 25 (14 credit points): *Maxwell theory*

Consider the Lagrange density of the electromagnetic field in the vacuum:

$$\mathcal{L} = -\frac{\sqrt{-g}}{16\pi} F_{\mu\nu} F^{\mu\nu}, \quad \text{where} \quad F_{\mu\nu} := 2\,\partial_{[\mu} A_{\nu]}\,.$$

25.1 Derive the field equations by means of the principle of least action.

25.2 Calculate the energy–momentum tensor $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$.

25.3 Show by direct calculation that the covariant divergence of the energy–momentum tensor $\nabla_\mu T^{\mu\nu}$ vanishes.

Exercise 26 (6 credit points): *Conformal transformations*

Two metrics g und $\bar{g}$ are defined to be *conformal* to each other if there is a non-vanishing differentiable function $\Omega(x)$ such that

$$\bar{g}_{\mu\nu}(x) = \Omega^2(x)\, g_{\mu\nu}(x)\,.$$

26.1 Show that angles between two vectors are conserved under a conformal transformation.

26.2 Check that the Christoffel symbol behaves under a conformal transformation as follows:

$$\bar{\Gamma}^\mu{}_{\nu\kappa} = \Gamma^\mu{}_{\nu\kappa} + S^\mu{}_{\nu\kappa}, \quad \text{where} \quad S^\mu{}_{\nu\kappa} := 2\,\delta^\mu_{(\nu}\sigma_{\kappa)} - g_{\nu\kappa}\,\sigma^\mu \quad \text{and} \quad \sigma_\mu := \partial_\mu \log\Omega\,.$$

Is $S^\mu{}_{\nu\kappa}$ a tensor?

26.3 Show that lightlike geodesics with respect to a metric $g_{\mu\nu}$ are also lightlike geodesics with respect to a conformally transformed metric $\bar{g}_{\mu\nu}$.