University of Cologne
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www.thp.uni-koeln.de/gravitation/courses/rci13.html

10th exercise sheet on Relativity and Cosmology I

Winter term 2013/14

Deadline for delivery: Thursday, 9th January 2014 during the exercise class.

Exercise 26 (14 credit points): *Maxwell theory*

Consider the Lagrange density of the electromagnetic field in the vacuum:

$$\mathcal{L} = -\frac{\sqrt{-g}}{16\pi} F_{\mu\nu} F^{\mu\nu}, \quad \text{where} \quad F_{\mu\nu} := 2\,\partial_{[\mu} A_{\nu]} \,.$$

26.1 Derive the field equations by means of the principle of least action.

26.2 Calculate the energy–momentum tensor $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$.

26.3 Show by direct calculation that the covariant divergence of the energy–momentum tensor $\nabla_\mu T^{\mu\nu}$ vanishes.

Exercise 27 (6 credit points): *Conformal transformations*

Two metrics g und $\bar{g}$ are defined to be *conformal* to each other if there is a non-vanishing differentiable function $\Omega(x)$ such that

$$\bar{g}_{\mu\nu}(x) = \Omega^2(x)\, g_{\mu\nu}(x) \,.$$

27.1 Show that angles between two vectors are conserved under a conformal transformation.

27.2 Check that the Christoffel symbol behaves under a conformal transformation as follows:

$$\bar{\Gamma}^{\mu}{}_{\nu\kappa} = \Gamma^{\mu}{}_{\nu\kappa} + S^{\mu}{}_{\nu\kappa}\,, \quad \text{where} \quad S^{\mu}{}_{\nu\kappa} := 2\,\delta^{\mu}_{(\nu}\sigma_{\kappa)} - g_{\nu\kappa}\,\sigma^{\mu} \quad \text{and} \quad \sigma_\mu := \partial_\mu \log\Omega \,.$$

Is $S^{\mu}{}_{\nu\kappa}$ a tensor?

27.3 Show that lightlike geodesics with respect to a metric $g_{\mu\nu}$ are also lightlike geodesics with respect to a conformally transformed metric $\bar{g}_{\mu\nu}$.