University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Branislav Nikolic and Patrick Wong

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www.thp.uni-koeln.de/gravitation/courses/rci1516.html

7th exercise sheet on Relativity and Cosmology I

Winter term 2015/16

Deadline for delivery: Thursday, 10th December 2015 during the exercise class.

Exercise 18 (10 credit points): *On the covariant derivative*

18.1 In order to define the general covariant derivative $\widetilde{\nabla}_\mu$, one only needs *a connection* $\widetilde{\Gamma}^{\nu}{}_{\mu\lambda}$, where at first the connection is not linked to the metric in any way:

$$\widetilde{\nabla}_\mu A^{\nu}{}_{\kappa} = \partial_\mu A^{\nu}{}_{\kappa} + \widetilde{\Gamma}^{\nu}{}_{\mu\lambda} A^{\lambda}{}_{\kappa} - \widetilde{\Gamma}^{\lambda}{}_{\mu\kappa} A^{\nu}{}_{\lambda}.$$

Show that the two conditions

$$Q_{\mu\nu\kappa} \equiv -\widetilde{\nabla}_\mu g_{\nu\kappa} = 0\,, \qquad T^{\mu}{}_{\nu\kappa} \equiv 2\,\widetilde{\Gamma}^{\mu}{}_{[\nu\kappa]} = 0$$

are equivalent to the condition that the connection $\widetilde{\Gamma}^{\mu}{}_{\nu\kappa}$ is the Christoffel symbol of second kind $\Gamma^{\mu}{}_{\nu\kappa}$. The quantities $Q_{\mu\nu\kappa}$ and $T^{\mu}{}_{\nu\kappa}$ are called *non-metricity* (see Exercise 17) and *torsion*, respectively. Are they tensors?

18.2 Let V^μ be a vector field and $\mathcal{V}^\mu \equiv \sqrt{-g}\, V^\mu$ be the corresponding vector density. Show that

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}\,V^\mu\big) \quad \text{and} \quad \nabla_\mu \mathcal{V}^\mu = \partial_\mu \mathcal{V}^\mu\,.$$

18.3 The covariant wave operator for a scalar field ϕ is given by

$$\Box\phi \equiv \nabla^\mu \nabla_\mu \phi\,.$$

Rewrite this by means of **18.2** such that the resulting expression only contains partial derivatives.

As an example, calculate the wave operator in 3-dimensional spherical coordinates.

Exercise 19 (5 credit points): *Riemannian normal coordinates*

Show that near the origin of a Riemannian normal coordinate system ($\xi^\mu \ll 1$) the following holds:

$$g_{\mu\nu}(0+\xi) = \eta_{\mu\nu} + \frac{1}{3}\,R_{\mu\kappa\lambda\nu}(0)\,\xi^\kappa\,\xi^\lambda + \ldots$$

Give a physical interpretation.

Exercise 20 (5 credit points): *Algebraic identities of Riemann tensor*

Show that, in Riemannian spaces, i.e. where the connection $\widetilde{\Gamma}^{\mu}{}_{\nu\kappa}$ is given by the Christoffel symbol of the second kind $\Gamma^{\mu}{}_{\nu\kappa}$, the Riemann tensor satisfies the following algebraic identities:

$$R_{\mu\nu(\alpha\beta)} = 0\,, \quad R_{(\mu\nu)\alpha\beta} = 0\,, \quad R_{\mu[\nu\alpha\beta]} = 0\,.$$

Can you give an illustrative interpretation of the first two identities?