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www.thp.uni-koeln.de/gravitation/courses/rcii13.html

# 1st exercise sheet on Relativity and Cosmology II

Summer term 2013

**Deadline for delivery:** Wednesday, 17th April 2013 at the end of the lecture.*

## Exercise 1 (9 credit points): *The Schwarzschild metric in isotropic coordinates*

Consider the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,.$$

**1.1** Use the coordinate transformation

$$t = \bar{t}\,, \qquad r = \left(1 + \frac{M}{2\bar{r}}\right)^2 \bar{r}$$

to express the metric in terms of the so-called *isotropic coordinates* $\bar{t}$, $\bar{r}$.

How does the metric behave at the horizon?

**1.2** Use the Schwarzschild geometry in isotropic coordinates derived above to calculate the surface of an equatorial circular ring that ranges from the Schwarzschild radius to a fixed radius $R$, as well as the volume of a spherical shell between these radii.

Compare your results to those in a Euclidean space.

## Exercise 2 (6 credit points): *Wormholes*

Consider the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + \left(b^2 + r^2\right)\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right),$$

where $b$ is a constant of dimension length. Illustrate this geometry by embedding it into a flat space.

To do so, choose the slicings $t = $ const. and $\vartheta = \frac{\pi}{2}$. Why does this suffice?
Map the resulting 2-dimensional geometry with the line element

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + \left(b^2 + r^2\right)\mathrm{d}\varphi^2$$

onto a surface in $\mathbb{R}^3$ having the same geometry. Use cylindrical coordinates with the line element

$$\mathrm{d}\ell^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2 + \mathrm{d}z^2\,.$$

Find the function $z(r(\rho))$ and draw a sketch of the rotation surface of the curve described by this function.

## Exercise 3 (5 credit points): *Lemaître coordinates*

Find a suitable coordinate transformation to show that the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \frac{4}{9}\left[\frac{9M}{2\,(r-t)}\right]^{\frac{2}{3}}\mathrm{d}r^2 + \left[\frac{9M}{2}(r-t)^2\right]^{\frac{2}{3}}\mathrm{d}\Omega^2\,,$$

which seems to be dynamical, is in fact the static Schwarzschild metric.

*This is an exception in order to make sure that we can discuss this exercise sheet already in the first exercise class on 18th April, because there will not be an exercise class on 25th April.