University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Manuel Krämer

19th June 2013

www.thp.uni-koeln.de/gravitation/courses/rcii13.html

10th exercise sheet on Relativity and Cosmology II

Summer term 2013

Deadline for delivery: Thursday, 27th June 2013 during the exercise class.

Exercise 23 (20 **bonus** points): *Derivation of the Friedmann equations*

The aim of this exercise is to derive the Friedmann equations using the Cartan formalism.

23.1 Start with the Friedmann–Lemaître–Robertson–Walker line element in the form

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right]$$

and find the orthonormal cobasis ϑ^μ to rewrite this line element as $\mathrm{d}s^2 = \eta_{\mu\nu}\,\vartheta^\mu \otimes \vartheta^\nu$.
For convenience, use the definition $w := \sqrt{1-kr^2}$.

23.2 Calculate the exterior derivatives $\mathrm{d}\vartheta^\mu$.

23.3 Determine the connection forms $\omega^\mu{}_\nu$.

Hint: Use the metricity condition as well as the first Cartan structure equation.

23.4 Calculate the curvature 2-forms $\Omega^\mu{}_\nu$ and deduce the components of the Riemann curvature tensor $R^\mu{}_{\nu\lambda\kappa}$ by means of the second Cartan structure equation.

23.5 Determine the components of the Ricci tensor $R_{\mu\nu}$ as well as the Ricci scalar R.

23.6 Calculate the components of the Einstein tensor $G^\mu{}_\nu$ and derive the Friedmann equations by using the Einstein equations with the energy–momentum tensor $T^\mu{}_\nu = \mathrm{diag}\,(-\rho(t), P(t), P(t), P(t))$, where ρ is the energy density and P the pressure of an ideal fluid filling the universe.

Why do we use mixed components here?