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3rd July 2013

www.thp.uni-koeln.de/gravitation/courses/rcii13.html

12th exercise sheet on Relativity and Cosmology II

Summer term 2013

Deadline for delivery: Wednesday, 10th July 2013 at the end of the lecture, since there will be **no** exercise class on 11th July.

Exercise 26 (12 credit points): *Dark energy*

One way to simulate a cosmological constant is by means of a homogeneous scalar field ϕ with a suitable potential $V(\phi)$. For this purpose, consider the action

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\,\left(\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right).$$

26.1 Derive the equation of motion for a homogeneous field $\phi(t)$ in a Friedmann universe.

26.2 Calculate the energy–momentum tensor of the scalar field by means of a variation with respect to the metric. Specialize this calculation to a homogeneous field in a Friedmann universe and identify its energy–momentum tensor with that of an ideal fluid. That way, determine the energy density ρ_ϕ and the pressure p_ϕ. For which idealization does ϕ describe a cosmological constant?

26.3 In a concrete model one considers the potential $V(\phi) = \kappa/\phi^\alpha$ with at first arbitrary parameters κ and α. The scale factor shall obey the time evolution $a(t) \propto t^n$ (universe with $k = 0$; $n = \frac{2}{3}$ during matter domination, $n = \frac{1}{2}$ during radiation domination).

Look for a solution for ϕ of the form $\phi(t) \propto t^A$. Determine A and find the relation that has to be imposed between κ and α. Finally, calculate the energy density ρ_ϕ and compare this to the density ρ of matter (or radiation, respectively).

Exercise 27 (8 credit points): *De Sitter space*

Show that the spacetime characterized by the *de Sitter metric*

$$\mathrm{d}s^2 = -\left(1-\frac{r^2}{\alpha^2}\right)\mathrm{d}t^2 + \left(1-\frac{r^2}{\alpha^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2\right), \qquad \alpha^2 = \frac{3}{\Lambda},$$

can be mapped isometrically onto the four-dimensional subspace

$$x^2 + y^2 + z^2 + u^2 - T^2 = \alpha^2$$

of $\mathbb{R}^5$ with metric

$$\mathrm{d}s^2 = -\,\mathrm{d}T^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}u^2\,.$$

Guidance: Replace the Euclidean coordinates (y, z, u) of the embedding space by the usual polar coordinates (r, θ, ϕ). Then consider the embedding $(t, r, \theta, \phi) \mapsto (T, x, r, \theta, \phi)$ with

$$(t, r) \mapsto (T, x)\,, \quad T = \sqrt{\alpha^2 - r^2}\,\sinh\left(\frac{t}{\alpha}\right), \quad x = \sqrt{\alpha^2 - r^2}\,\cosh\left(\frac{t}{\alpha}\right),$$

whereas the remaining coordinates are unchanged.

Draw a (T, x) diagram and indicate the curves $t = \text{const.}$ as well as $r = \text{const.}$

Remark: This transformation is restricted to $x \geq |T|$, $x > 0$. However, one can cover the whole range of coordinates analogously to the Kruskal diagram by means of additional transformations.