Addendum: The Einstein field equation in exterior calculus

file GR_Lecture_note1.tex, 01 July 2014, Huan-Hsin Tseng, with corr.
University of Cologne, General Relativity Lecture on 24 April 2014 by fwh

Given a 4-dimensional manifold M with coframe ϑ^μ, (dual) frame e_μ, metric tensor $g(\cdot,\cdot)$, covariant exterior derivative D, and linear connection 1-form on M, namely $\omega^\mu{}_\nu = L^\mu{}_{\nu\rho}\,\vartheta^\rho$, where $L^\mu{}_{\nu\rho}$ are the components of the linear connection.

To derive the Einstein equation, we assume a Riemannian spacetime (M, g, D), for which D (Levi-Civita connection) fulfills the to two conditions of vanishing nonmetricity

$$Q_{\mu\nu} := Dg_{\mu\nu} = Q_{\mu\nu\lambda}\,\vartheta^\lambda = 0 \tag{1}$$

and of vanishing torsion

$$\Theta^\mu := D\vartheta^\mu = d\vartheta^\mu + \omega^\mu{}_\nu \wedge \vartheta^\nu = \frac{1}{2}T^\mu{}_{\nu\lambda}\,\vartheta^\nu \wedge \vartheta^\lambda = 0\,; \tag{2}$$

this constraint is also called the 1^{st} Cartan structure equation for a Riemannian space. Moreover, we have the curvature 2-forms

$$\Omega^\mu{}_\nu := d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu = \frac{1}{2}R^\mu{}_{\nu\lambda\rho}\,\vartheta^\lambda \wedge \vartheta^\rho\,. \tag{3}$$

We turn now to the Bianchi type identities by differentating successively nonmetricity, torsion, and curvature, respectively:

$$\begin{aligned}
DQ_{\mu\nu} &= DDg_{\mu\nu} \overset{\text{Ricci lemma}}{=} -\Omega^\alpha{}_\mu\, g_{\alpha\nu} - \Omega^\beta{}_\nu\, g_{\mu\beta} = -\Omega_{\nu\mu} - \Omega_{\mu\nu} = -2\Omega_{(\mu\nu)} \overset{!}{=} 0\,, && (4)\\
D\Theta^\mu &= DD\vartheta^\mu \overset{\text{Ricci lemma}}{=} \Omega^\mu{}_\nu \wedge \vartheta^\nu = \frac{1}{2}R^\mu{}_{[\nu\rho\sigma]}\,\vartheta^\rho \wedge \vartheta^\sigma \wedge \vartheta^\nu \overset{!}{=} 0\,, && (5)\\
D\Omega^\mu{}_\nu &= D\,\text{``}D\omega^\mu{}_\nu\text{''} = D\left(d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu\right) = d(d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu) && (6)\\
&\qquad + \omega^\mu{}_\beta \wedge (\cdots) - \omega^\gamma{}_\nu(\cdots) = \cdots = 0\,,
\end{aligned}$$

where $\overset{!}{=}$ we refer back to our two postulates (1) or (2), respectively. In components, Eqs.(4) and (5) can also be written as

$$R_{(\mu\nu)\rho\sigma} = 0 \tag{7}$$

and

$$R^\mu{}_{[\nu\rho\sigma]} = 0 \qquad \text{or} \quad R_{[\mu\nu\rho\sigma]} = 0\,; \tag{8}$$

the last equation follows because of (7). Eq.(5), $\Omega^\mu{}_\nu \wedge \vartheta^\nu = 0$, is called the 1^{st} Bianchi identity and Eq.(6), $D\Omega^\mu{}_\nu = 0$, the 2^{nd} Bianchi identity of a Riemannian space.

The Hodge star operator is a tool for mapping p-forms into $(n-p)$-forms. Take a p-form Ψ, decomposing as

$$\Psi = \frac{1}{p!}\Psi_{\gamma_1\cdots\gamma_p}\,\underbrace{\vartheta^{\gamma_1} \wedge \vartheta^{\gamma_2} \wedge \cdots \wedge \vartheta^{\gamma_p}}_{\vartheta^{\gamma_1\cdots\gamma_p}}\,, \tag{9}$$

where we define $\vartheta^{\gamma_1\cdots\gamma_p} := \vartheta^{\gamma_1} \wedge \vartheta^{\gamma_2} \wedge \cdots \wedge \vartheta^{\gamma_p}$. Then we can construct a $(n-p)$-form by the Hodge dual map $\star : \Lambda^p(M) \to \Lambda^{(n-p)}(M)$,

$$\star\,\Psi := \frac{1}{(n-p)!p!}\sqrt{|\det g_{\mu\nu}|}\,\epsilon_{\alpha_1\alpha_2\cdots\alpha_p\beta_1\cdots\beta_{n-p}}\,g^{\alpha_1\gamma_1}\cdots g^{\alpha_p\gamma_p}\cdot\Psi_{\gamma_1\cdots\gamma_p}\,\vartheta^{\beta_1\cdots\beta_{n-p}} \tag{10}$$

where $\epsilon_{\alpha_1\alpha_2\cdots\alpha_p\beta_1\cdots\beta_{n-p}}$ is the totally antisymmetric Levi-Civita symbol with value $+1$ for *even* permutations, -1 for *odd* permutations, and 0 otherwise. It is a *tensor density*, and $\varepsilon_{\alpha_1\cdots\beta_{n-p}} := \sqrt{|det g_{\mu\nu}|}\,\epsilon_{\alpha_1\alpha_2\cdots\alpha_p\beta_1\cdots\beta_{n-p}}$ is a *unit tensor*.

There are 3 important formula for the Hodge star $\star$. For $\Phi, \Psi \in \Lambda^p(M)$,

$$\begin{aligned} \star\,\Phi \wedge \Psi &= \star\Psi \wedge \Phi\,, & (11)\\ \vartheta^\mu \wedge \left(e_\mu \rfloor \Phi\right) &= p\,\Phi\,, & (12)\\ \star\left(\Phi \wedge \vartheta_\mu\right) &= e_\mu \rfloor \star\Phi\,. & (13) \end{aligned}$$

For the action principle, we need the variation of the $\star$ (since $\star$ depends on ϑ^μ and $g_{\mu\nu}$). We assume from now on *orthonormal frames*, which we can always choose. We take a formula from the literature (see [1], for example),

$$(\delta\cdot\star - \star\cdot\delta)\,\phi = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \star\phi) - \star\left[\delta\vartheta^\alpha \wedge (e_\alpha \rfloor \phi)\right]. \tag{14}$$

For the decomposition of arbitrary forms in 4d, it is useful to define the following ϑ-basis:

$$1, \quad \vartheta^\mu, \quad \vartheta^{\mu\nu}, \quad \vartheta^{\mu\nu\lambda}, \quad \vartheta^{\mu\nu\lambda\rho}, \tag{15}$$

which are a 0-form, 1-forms, 2-forms, 3-forms, and 4-forms respectively. Some examples are

- the *electromagnetic potential* $A = A_\mu\,\vartheta^\mu = A_i dx^i$,
- the *electromagnetic field strength* $F = \frac{1}{2}F_{\mu\nu}\,\vartheta^{\mu\nu} = \frac{1}{2}F_{ij}dx^i \wedge dx^j = E \wedge dt + B$, and
- the *electric current density* as the 3-form $\mathfrak{J} = \frac{1}{3!}\mathfrak{J}_{\mu\nu\rho}\,\vartheta^{\mu\nu\rho} = \frac{1}{3!}\mathfrak{J}_{ijk}dx^i \wedge dx^j \wedge dx^k$.

In GR, we have the connection $\omega^\mu{}_\nu$ as 1-form, the curvature $\Omega^\mu{}_\nu$ as 2-form, and the energy-momentum density of matter $\mathfrak{T}_\mu$ as (covector-valued) 3-form.

The Maxwell vacuum field equations read

$$d\star F = \mathfrak{J}\,, \qquad dF = 0\,. \tag{16}$$

They are valid in this form for flat and for curved space, coframe and connection do not enter these equations; only the (conformal part of) the metric is felt by the Hodge star. They are diffeomorphism invariant equations that do neither depend on coordinates nor on frames. Via the Lorentz force density $\mathfrak{f}_\nu = (e_\mu \rfloor F) \wedge \mathfrak{J}$, one finds the energy-momentum density 3-form as

$$\mathfrak{T}_\mu^{\text{Max}} = \frac{1}{2}\left[F \wedge (e_\mu \rfloor \star F) - \star F \wedge (e_\mu \rfloor F)\right]. \tag{17}$$

Having the ϑ-basis (15), we can introduce the dual ε-basis for forms in 4-dimensional manifold. We start with the volume n-form:

$$\varepsilon := \sqrt{|\det g_{\mu\nu}|}\,\vartheta^{1\cdots n} = \frac{1}{n!}\sqrt{|\det g_{\mu\nu}|}\,\epsilon_{\mu_1\cdots\mu_n}\,\vartheta^{\mu_1\cdots\mu_n} \tag{18}$$

We now define successively the lower forms

$$\begin{aligned} \varepsilon_\mu &:= e_\mu\rfloor\varepsilon = \star\vartheta_\mu \qquad \text{(3-form)}\,,\\ \varepsilon_{\mu\nu} &:= e_\nu\rfloor\varepsilon_\mu = \star\vartheta_{\mu\nu} \qquad \text{(2-form)}\,,\\ \varepsilon_{\mu\nu\lambda} &:= e_\lambda\rfloor\varepsilon_{\mu\nu} = \star\vartheta_{\mu\nu\lambda} \qquad \text{(1-form)}\,,\\ \varepsilon_{\mu\nu\lambda\rho} &:= e_\rho\rfloor\varepsilon_{\mu\nu\lambda} = \star\vartheta_{\mu\nu\lambda\rho} \qquad \text{(0-form)}\,, \end{aligned} \tag{19}$$

where we recall $e_\mu\rfloor\vartheta^\nu = \delta^\nu_\mu$. In Riemannian space ($Q_{\mu\nu} = 0$, $\Theta^\mu = 0$), we have the following identities:

$$D\varepsilon = 0, \quad D\varepsilon_\mu = 0, \quad D\varepsilon_{\mu\nu} = 0, \quad D\varepsilon_{\mu\nu\lambda} = 0, \quad D\varepsilon_{\mu\nu\lambda\rho} = 0\,. \tag{20}$$

For the action principle, we need variations of $\delta\vartheta^\mu$, $\delta\omega^{\mu\nu}$ (not independent in the Riemannian case!). Rules:

$$\delta(\omega_1\wedge\omega_2) = \delta\omega_1\wedge\omega_2 + \omega_1\wedge\delta\omega_2, \tag{21}$$

for all $\omega_1\in\Lambda^p(M)$, $\omega_2\in\Lambda^q(M)$. Notice δ does not change sign across any rank, so it is *even*. Furthermore,

$$[d,\delta] = 0, \qquad \text{(but not valid for } D!)\,. \tag{22}$$

The variation of $\star$ is evaluated in (14).

The Lagrangian 4-form for electrodynamics is $L_{\text{elec}} \sim \star F\wedge F$. The $\star$ is required for parity reasons. A similar structure is manifest in the Hilbert-Einstein Lagrangian of general relativity:

$$\begin{aligned} V_{\text{HE}} &\sim \star\left(\vartheta_\mu\wedge\vartheta_\nu\right)\wedge\Omega^{\mu\nu} = \varepsilon_{\mu\nu}\wedge\Omega^{\mu\nu}\,,\\ V_{\text{HE}} &\overset{\text{(LL)}}{=} -\frac{1}{2\kappa}\varepsilon_{\mu\nu}\wedge\Omega^{\mu\nu}\,. \end{aligned} \tag{23}$$

Using orthonormal frames, we now vary $\delta\vartheta^\mu$, $\delta\omega^{\mu\nu}$ (dependent on $\delta\vartheta$ due to Levi-Civita connection of GR). By (21), we have

$$\delta V_{\text{HE}} = -\frac{1}{2\kappa}\left[\delta\varepsilon_{\mu\nu}\wedge\Omega^{\mu\nu} + \varepsilon_{\mu\nu}\wedge\delta\Omega^{\mu\nu}\right]\,, \tag{24}$$

where

$$\begin{aligned} \delta\varepsilon_{\mu\nu} = \delta\star\vartheta_{\mu\nu} &\overset{(14)}{=} \star\delta\vartheta_{\mu\nu} + \delta\vartheta^\alpha\wedge\left(e_\alpha\rfloor\varepsilon_{\mu\nu}\right) - \star\left[\delta\vartheta^\alpha\wedge\left(e_\alpha\rfloor\vartheta_{\mu\nu}\right)\right]\\ &= \star\left(\cancel{\delta\vartheta_\mu\wedge\vartheta_\nu} + \cancel{\vartheta_\mu\wedge\delta\vartheta_\nu}\right) + \delta\vartheta^\alpha\wedge\varepsilon_{\mu\nu\alpha} - \star\left[\delta\vartheta^\alpha\wedge\left(\cancel{g_{\alpha\mu}\vartheta_\nu} - \cancel{g_{\alpha\nu}\vartheta_\mu}\right)\right]\\ &= \delta\vartheta^\alpha\wedge\varepsilon_{\mu\nu\alpha}\,. \end{aligned} \tag{25}$$

The variation of $\Omega^{\mu\nu}$ yields

$$\delta\Omega^{\mu\nu} = d\delta\omega^{\mu\nu} + (\delta\omega^{\mu\alpha}) \wedge \omega_\alpha{}^\nu + \omega^{\mu\alpha} \wedge \delta\omega_\alpha{}^\nu \,. \tag{26}$$

We recall that the variation $\delta\omega^{\mu\nu}$ is *tensorial*, in contrast to the connection 1-form $\omega^{\mu\nu}$. Thus, we can take its covariant exterior derivative

$$D\delta\omega^{\mu\nu} = d\delta\omega^{\mu\nu} + \omega^\mu{}_\alpha \wedge \delta\omega^{\alpha\nu} + \omega^\nu{}_\alpha \wedge \delta\omega^{\mu\alpha} \,. \tag{27}$$

If we resolve this equation with respect to $d\delta\omega^{\mu\nu}$, substitute it in (26), and use the rule $[d, \delta] = 0$, we find straightforwardly

$$\delta\Omega^{\mu\nu} = D\delta\omega^{\mu\nu} \,, \tag{28}$$

We substitute (25) and (28) into (24) and add the matter Lagrangian,

$$\delta V_{\text{tot}} := \delta V_{\text{HE}} + \delta V_{\text{mat}} = -\frac{1}{2\kappa}\left[\delta\vartheta^\alpha \wedge \varepsilon_{\mu\nu\alpha} \wedge \Omega^{\mu\nu} + \varepsilon_{\mu\nu} \wedge D\delta\omega^{\mu\nu}\right] + \delta\vartheta^\alpha \wedge \mathfrak{T}_\alpha \tag{29}$$

where $\mathfrak{T}_\alpha := \delta L_{\text{mat}}/\delta\vartheta^\alpha$ denotes the energy-momentum 3-form of matter. Taking into account (20), this can be rewritten as

$$\delta V_{\text{tot}} = -\frac{1}{2\kappa}\left[\delta\vartheta^\alpha \wedge \left(\varepsilon_{\mu\nu\alpha} \wedge \Omega^{\mu\nu} - 2\kappa\mathfrak{T}_\alpha\right) + d\left(\varepsilon_{\mu\nu} \wedge \delta\omega^{\mu\nu}\right)\right] \tag{30}$$

The variational principle requires $\delta V_{\text{tot}} = 0$ for all $\delta\vartheta^\alpha$. Since the surface term does not contribute, we arrive at

$$\boxed{\frac{1}{2}\varepsilon_{\mu\nu\lambda} \wedge \Omega^{\nu\lambda} = \kappa\,\mathfrak{T}_\mu.} \tag{31}$$

This is Einstein's equation in exterior calculus, with the Einstein 3-form $G_\mu := \frac{1}{2}\varepsilon_{\mu\nu\lambda} \wedge \Omega^{\nu\lambda}$. One can also include the cosmological term into (31) by adding $\Lambda\varepsilon_\mu$, with the cosmological constant $\Lambda \in \mathbb{R}$.

The translation to tensor calculus, one develops the Einstein and the matter energy-momentum 3-forms with respect to the 3-form basis ε^ν, namely $G_\mu = \widetilde{G}_{\mu\nu}\,\varepsilon^\nu$ and $\mathfrak{T}_\mu = T_{\mu\nu}\,\varepsilon^\nu$. In this way one can recover the Einstein equation $\widetilde{G}_{\mu\nu} = \kappa\,T_{\mu\nu}$ in tensor calculus.

Incidentally, GR can be generalized to the so-called Einstein-Cartan theory of gravitation (EC) by allowing the occurrence of a nonvanishing torsion, that is, the assumption (2) will be dropped. Then the variation $\delta\omega^{\mu\nu}$ becomes independent of $\delta\vartheta^\mu$ and the varied matter Lagrangian picks up an additional spin-dependent piece $+\delta\omega^{\mu\nu} \wedge \mathfrak{S}_{\mu\nu}$. The variation of $\delta\vartheta^\mu$ is exactly as above; only the new relation $D\varepsilon_{\mu\nu} = \Theta^\alpha \wedge \varepsilon_{\alpha\mu\nu} \neq 0$ makes a difference and provides a new term also in the gravitational Lagrangain, see Kibble [3].

Sample program for the computer algebra system Reduce with the Excalc package:

Coframe `o`, frame `e`, ranks of forms declared by `pform`, exterior product sign $\wedge$, interior product sign for a vector with a form $|\_$, Riemannian connection 1-form `riemannconx` = `chris1`, curvature 2-form `curv2`, Einstein 3-form `ein3`, Hodge star operator #.

```
% file Sun/ueb2.exi

% Reissner-Nordstrom-deSitter

load excalc$

pform psi=0$
fdomain psi=psi(r)$

coframe o(t)     = psi*d t,
        o(r)     = (1/psi)*d r,
        o(theta) = r*d theta,
        o(phi)   = r*sin(theta)*d phi with
        signature(-1,1,1,1)$

frame e$
psi := sqrt(1-2*m/r+k*(q/r)**2+lam*r**2/3)$

riemannconx chris1$

chris1(a,b):=chris1(a,b);

pform curv2(a,b)=2, ein3(a)=3$ antisymmetric curv2$

curv2(a,b):=d chris1(a,b)+ chris1(a,-c)^chris1(c,b);
ein3(a)   := (1/2)* #(o(a)^o(b)^o(c)) ^ curv2(-b,-c);

end$
```

Description of this and other programs in Stauffer et al. [2].

Bibliography

[1] U. Muench et al., *A Small guide to variations in teleparallel gauge theories of gravity and the Kaniel-Itin model*, Gen. Rel. Grav. **30**, 933 (1998) [arXiv:gr-qc/9801036]; is in the Physics Library.

[2] D. Stauffer, F. W. Hehl, N. Ito, V. Winkelmann, and J. G. Zabolitzky, *Computer Simulation and Computer Algebra. Lectures for Beginners*, 3rd ed., Springer, Berlin (1993); is in the Physics Library.

[3] T. W. B. Kibble, *Lorentz invariance and the gravitational field*, J. Math. Phys. **2**, 212 (1961); is in the Physics Library.