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22nd April 2014

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3rd exercise sheet on Relativity and Cosmology II

Summer term 2014

Deadline for delivery: Wednesday, 30th April 2014 during the exercise class.

Exercise 5 (10 credit points): *Differential forms*

5.1 Consider an n-dimensional manifold with a metric. Let $\{\omega^i\}$ be an orthonormal basis of 1-forms, and let ω be the preferred volume form $\omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n$.

Show that in an arbitrary coordinate system $\{x^k\}$ the following holds:

$$\omega = \sqrt{|g|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n\,, \tag{1}$$

where g denotes the determinant of the metric whose components g_{ij} are given in these coordinates.

5.2 The contraction of a p-form ω (with components $\omega_{ij\ldots k}$) with a vector v (with components v^i) is given by $[\omega(v)]_{j\ldots k} = \omega_{ij\ldots k}\, v^i$. Consider the n-form $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$.

Show that with a given vector field v the following holds:

$$\mathrm{d}[\omega(v)] = v^i{}_{,i}\, \omega\,. \tag{2}$$

5.3 We define $(\mathrm{div}_\omega v)\, \omega := \mathrm{d}[\omega(v)]$.

Show that by using coordinates in which ω has the form $\omega = f\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$ the following holds:

$$\mathrm{div}_\omega v = \frac{1}{f} \left(f v^i \right)_{,i}\,. \tag{3}$$

5.4 In three-dimensional Euclidean space, the preferred volume form is given by $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Show that in spherical coordinates this volume form is given by $\omega = r^2 \sin\theta\, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$.
Use the result of **5.3** to show that the divergence of a vector field

$$v = v^r \frac{\partial}{\partial r} + v^\theta \frac{\partial}{\partial \theta} + v^\phi \frac{\partial}{\partial \phi} \tag{4}$$

is given by

$$\mathrm{div}\, v = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 v^r \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta\, v^\theta \right) + \frac{\partial v^\phi}{\partial \phi}\,. \tag{5}$$

Exercise 6 (10 credit points): *Electrodynamics in flat spacetime*

Differential forms are a convenient tool for field theories, as we will show in this exercise. Consider electrodynamics in flat spacetime. The Faraday 2-form $\mathbf{F}$ describing an arbitrary electromagnetic field is given by

$$\mathbf{F} := \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -E_x\, \mathrm{d}t \wedge \mathrm{d}x - E_y\, \mathrm{d}t \wedge \mathrm{d}y - E_z\, \mathrm{d}t \wedge \mathrm{d}z + B_x\, \mathrm{d}y \wedge \mathrm{d}z + B_y\, \mathrm{d}z \wedge \mathrm{d}x + B_z\, \mathrm{d}x \wedge \mathrm{d}y\,, \tag{6}$$

and the current 1-form is given by

$$\mathbf{j} := \rho\, \mathrm{d}t + j_x\, \mathrm{d}x + j_y\, \mathrm{d}y + j_z\, \mathrm{d}z\,. \tag{7}$$

See reverse.

6.1 The Hodge star operator $\star$ maps p-forms to $(4-p)$-forms. Therefore, 2-forms are mapped to 2-forms by this operator. The 2-form dual to the Faraday 2-form is the Maxwell 2-form, defined by $\tilde{\mathbf{F}} := \star\mathbf{F}$.

Use the following definition of the Hodge star acting on the 1-form basis

$$\star(\mathrm{d}t \wedge \mathrm{d}x) := \mathrm{d}y \wedge \mathrm{d}z \quad \text{(and cyclic permutations)} \tag{8}$$

and show that the Maxwell 2-form is given by

$$\tilde{\mathbf{F}} = -\,B_x\,\mathrm{d}t \wedge \mathrm{d}x - B_y\,\mathrm{d}t \wedge \mathrm{d}y - B_z\,\mathrm{d}t \wedge \mathrm{d}z + E_x\,\mathrm{d}y \wedge \mathrm{d}z + E_y\,\mathrm{d}z \wedge \mathrm{d}x + E_z\,\mathrm{d}x \wedge \mathrm{d}y\,. \tag{9}$$

6.2 Show that the equation $\mathrm{d}\mathbf{F} = 0$ corresponds to the two homogeneous Maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{E} = -\,\partial_t \vec{B}\,. \tag{10}$$

6.3 Calculate $\star\mathbf{j}$ (a 3-form), which is the dual of the current 1-form $\mathbf{j}$. For that, use the relation

$$\star\mathrm{d}t := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \quad \text{(and cyclic permutations)}\,. \tag{11}$$

Then show that the equation $\mathrm{d}\tilde{\mathbf{F}} = 4\pi \star \mathbf{j}$ corresponds to the two inhomogeneous Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad \text{and} \quad \vec{\nabla} \times \vec{B} = \partial_t \vec{E} + 4\pi\vec{j}\,. \tag{12}$$

6.4 The relation $\mathrm{d}(\mathrm{d}\boldsymbol{\omega}) = 0$ holds for any p-form $\boldsymbol{\omega}$.

Choosing $\boldsymbol{\omega} = \star\mathbf{j}$, show that this yields the continuity equation $\vec{\nabla} \cdot \vec{j} + \partial_t \rho = 0$.

These calculations have been carried out using the exterior derivative d. In curved spacetime, the covariant derivative D has to be used instead to accommodate for the curvature of spacetime.

6.5 Explain in your own words why a covariant derivative is needed on a curved background, and give special attention to the connection. What is the intuitive interpretation of the connection? Why can it be chosen to be zero in Minkowski space?