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www.thp.uni-koeln.de/gravitation/courses/rcii14.html

6th exercise sheet on Relativity and Cosmology II

Summer term 2014

Deadline for delivery: Wednesday, 21st May 2014 during the exercise class.

Exercise 10 (5 credit points): *Kruskal coordinates*

Derive the line element of the Schwarzschild metric in Kruskal coordinates as given in the lecture.

For this purpose, introduce a new radial coordinate (for $r > 2M$) as follows

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right).$$

Then perform the coordinate transformation:

$$X = \exp\left(\frac{r_*}{4M}\right) \cosh\left(\frac{t}{4M}\right), \quad T = \exp\left(\frac{r_*}{4M}\right) \sinh\left(\frac{t}{4M}\right).$$

Exercise 11 (6 credit points): *Another coordinate system*

Construct a coordinate system for the Schwarzschild metric that is singularity-free at the event horizon by transforming the Schwarzschild time t according to

$$t \to T = t + f(r).$$

Determine $f(r)$ by imposing that the prefactor of $\mathrm{d}r^2$ is equal to $+1$ in the transformed line element. Write out the transformed line element. Is it still static? Which parts of the Kruskal diagram are covered by these coordinates?

Exercise 12 (9 credit points): *Penrose diagrams*

12.1 Express the line element for Minkowski spacetime in terms of spherical coordinates (t, r, θ, ϕ). Then perform a coordinate transformation

$$u = t - r, \quad v = t + r.$$

Write out the transformed line element. How can one interpret the coordinates u and v?

12.2 Perform another coordinate transformation $(u, v) \mapsto (u', v')$ according to

$$u' = \arctan(u) =: t' - r', \quad v' = \arctan(v) =: t' + r'.$$

Draw a (t', r') diagram and hatch the area covered by these coordinates. Then draw a radial light ray in this diagram that goes from infinity (in the original coordinates) to $r = 0$ and back to infinity.
In a second (t', r') diagram, sketch the areas $t =$ const. and $r =$ const.

12.3 Calculate the line element in the primed coordinates and show that it is conformal to the line element

$$\mathrm{d}\bar{s}^2 = -4\left(\mathrm{d}t'^2 - \mathrm{d}r'^2\right) + \sin^2(2r')\,\mathrm{d}\Omega^2.$$