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www.thp.uni-koeln.de/gravitation/courses/rcii16.html

2nd exercise sheet on Relativity and Cosmology II

Summer term 2016

Deadline for delivery: Thursday, 28th April 2016 during the exercise class.

Exercise 3 (10 credit points): *Differential forms*

3.1 Consider a n-dimensional manifold with a metric. Let $\{\omega^i\}$ be an orthonormal basis of 1-forms, and let ω be the preferred volume form $\omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n$.

Show that in an arbitrary coordinate system $\{x^k\}$ the following holds:

$$\omega = \sqrt{|g|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n\,, \tag{1}$$

where g denotes the determinant of the metric whose components g_{ij} are given in these coordinates.

3.2 The contraction of a p-form ω (with components $\omega_{ij\ldots k}$) with a vector v (with components v^i) is given by $[\omega(v)]_{j\ldots k} = \omega_{ij\ldots k}\, v^i$. Consider the n-form $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$.

Show that with a given vector field v the following holds:

$$\mathrm{d}[\omega(v)] = v^i{}_{,i}\, \omega\,. \tag{2}$$

3.3 We define $(\mathrm{div}_\omega v)\, \omega := \mathrm{d}[\omega(v)]$.

Show that by using coordinates in which ω has the form $\omega = f\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$ the following holds:

$$\mathrm{div}_\omega v = \frac{1}{f} \left(f v^i\right)_{,i}\,. \tag{3}$$

3.4 In three-dimensional Euclidean space, the preferred volume form is given by $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Show that in spherical coordinates this volume form is given by $\omega = r^2 \sin\theta\, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$.
Use the result of **3.3** to show that the divergence of a vector field

$$v = v^r \frac{\partial}{\partial r} + v^\theta \frac{\partial}{\partial \theta} + v^\phi \frac{\partial}{\partial \phi} \tag{4}$$

is given by

$$\mathrm{div}\, v = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 v^r\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta\, v^\theta\right) + \frac{\partial v^\phi}{\partial \phi}\,. \tag{5}$$

Exercise 4 (10 credit points): *Electrodynamics in flat spacetime*

Differential forms are a convenient tool for field theories, as we will show in this exercise on the example of Maxwell electrodynamics in Minkowski spacetime. We know that electromagnetic field strength is given by Faraday antisymmetric tensor $F_{\mu\nu}$, and the current is given by j^μ. In the language of exterior calculus, $F_{\mu\nu}$ are components of the Faraday 2-form $\mathbf{F}$ describing an arbitrary electromagnetic field, given by

$$\mathbf{F} := \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -E_x\, \mathrm{d}t \wedge \mathrm{d}x - E_y\, \mathrm{d}t \wedge \mathrm{d}y - E_z\, \mathrm{d}t \wedge \mathrm{d}z + B_x\, \mathrm{d}y \wedge \mathrm{d}z + B_y\, \mathrm{d}z \wedge \mathrm{d}x + B_z\, \mathrm{d}x \wedge \mathrm{d}y\,, \tag{6}$$

while j^μ are components of the current 1-form $\mathbf{j}$ is given by

$$\mathbf{j} := j_\mu dx^\mu = -\rho\, \mathrm{d}t + j_x\, \mathrm{d}x + j_y\, \mathrm{d}y + j_z\, \mathrm{d}z\,. \tag{7}$$

See reverse.

4.1 The Hodge star operator $\star$ maps p-forms to $(4-p)$-forms. Therefore, 2-forms are mapped to 2-forms by this operator. The 2-form dual to the Faraday 2-form is the Maxwell 2-form, defined by $\mathbf{G} := \star\mathbf{F}$.

The Hodge star operator acts on the 2-form basis as follows

$$\star(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) := \frac{1}{2}\,\eta^{\mu\alpha}\eta^{\nu\beta}\varepsilon_{\alpha\beta\gamma\delta}\mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta\,, \tag{8}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the inverse Minkowski metric and $\varepsilon_{\alpha\beta\gamma\delta}$ totally symmetric Levi-Civita tensor density ($\varepsilon_{0123} = +1$). For example,

$$\star(\mathrm{d}t \wedge \mathrm{d}x) := -\mathrm{d}y \wedge \mathrm{d}z\,, \quad \text{etc.} \tag{9}$$

Show that the Maxwell 2-form is given by

$$\mathbf{G} = B_x\,\mathrm{d}t \wedge \mathrm{d}x + B_y\,\mathrm{d}t \wedge \mathrm{d}y + B_z\,\mathrm{d}t \wedge \mathrm{d}z + E_x\,\mathrm{d}y \wedge \mathrm{d}z + E_y\,\mathrm{d}z \wedge \mathrm{d}x + E_z\,\mathrm{d}x \wedge \mathrm{d}y\,. \tag{10}$$

Which relation between the components of $\mathbf{F}$ and $\mathbf{G}$ holds?

4.2 With these definitions, the Maxwell equations can be written in a compact form: $\mathrm{d}\mathbf{F} = 0$ and $\mathrm{d}\mathbf{G} = 4\pi \star \mathbf{j}$, therefore,

a) Show that the equation $\mathrm{d}\mathbf{F} = 0$ corresponds to the two homogeneous Maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0\,. \tag{11}$$

b) Calculate $\star\mathbf{j}$ (a 3-form), which is the dual of the current 1-form $\mathbf{j}$. For that, use the following relations

$$\star\mathrm{d}x^\mu := \frac{1}{3!}\eta^{\mu\alpha}\varepsilon_{\alpha\beta\gamma\delta}\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta\,, \tag{12}$$

for example

$$\star\mathrm{d}t := -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,, \quad \text{etc.} \tag{13}$$

c) Then show that the equation $\mathrm{d}\mathbf{G} = 4\pi \star \mathbf{j}$ corresponds to the two inhomogeneous Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad \text{and} \quad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 4\pi\vec{j}\,. \tag{14}$$

d) What can you do to turn (11) into (14) in the vacuum case?

4.3 The exterior derivative is nilpotent, i.e. the relation $\mathrm{d}(\mathrm{d}\boldsymbol{\omega}) = 0$ holds for any p-form $\boldsymbol{\omega}$.

Choosing $\boldsymbol{\omega} = \star\mathbf{j}$, show that this yields the continuity equation $\partial_t\rho - \vec{\nabla} \cdot \vec{j} = 0$.

These calculations have been carried out using the exterior derivative d. In curved spacetime, the covariant derivative D has to be used instead to accommodate for the curvature of spacetime.

4.4 Explain in your own words why a covariant derivative is needed on a curved background, and give special attention to the connection. What is the intuitive interpretation of the connection? Why can it be chosen to be zero in Minkowski space?