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# 3rd exercise sheet on Relativity and Cosmology II

Summer term 2016

**Deadline for delivery: Friday, 6th May 2016 at the end of the lecture.**

## Exercise 5 (20 credit points): *Derivation of the Schwarzschild solution in Cartan calculus*

The aim of this exercise is to derive the Schwarzschild solution using the Cartan formalism.

Due to the symmetry of Schwarzschild spacetime, we can start with the following ansatz in coordinates:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\,\mathrm{e}^{2a(r,t)}\,\mathrm{d}t^2 + \mathrm{e}^{2b(r,t)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\,. \tag{1}$$

In terms of the pseudo-orthogonal coframe basis $\{\vartheta^i\}$, $i = 0,\ldots,3$, the metric takes the form

$$\mathrm{d}s^2 = \eta_{ij}\,\vartheta^i \otimes \vartheta^j = -\,\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3\,. \tag{2}$$

In this exercise, Latin letters are used for anholonomic indices, whereas Greek letters are used for coordinate indices. Holonomic coordinate indices and anholonomic frame indices can be converted into each other by using the tetrad defined via

$$\vartheta^i = e_\mu{}^i\,\mathrm{d}x^\mu\,. \tag{3}$$

**5.1** Show that the tetrad components are given by

$$\left(e_\mu{}^i\right) = \mathrm{diag}\left(\mathrm{e}^{a(r,t)},\, \mathrm{e}^{b(r,t)},\, r,\, r\sin\theta\right) \tag{4}$$

or, equivalently, that

$$\vartheta^t = \mathrm{e}^{a(r,t)}\mathrm{d}t\,,\quad \vartheta^r = \mathrm{e}^{b(r,t)}\,\mathrm{d}r\,,\quad \vartheta^\theta = r\,\mathrm{d}\theta\,,\quad \vartheta^\phi = r\sin\theta\,\mathrm{d}\phi\,. \tag{5}$$

**5.2** Now, calculate the exterior derivatives $\mathrm{d}\vartheta^i$. Insert these into the first Cartan structure equation

$$\mathrm{d}\vartheta^i + \omega^i{}_j \wedge \vartheta^j = 0 \tag{6}$$

to determine the 1-form-valued components $\omega^i{}_j$ of the connection.

*Intermediate result:* These components read

$$\omega^r{}_t = a'\,\mathrm{e}^{-b(t,r)}\,\vartheta^t + \dot{b}\,\mathrm{e}^{-a(t,r)}\,\vartheta^r\,, \qquad \omega^\phi{}_r = \frac{\mathrm{e}^{-b(t,r)}}{r}\,\vartheta^\phi\,, \qquad \omega^\phi{}_\theta = \frac{\cot\theta}{r}\,\vartheta^\phi\,, \tag{7}$$

$$\omega^\theta{}_r = \frac{\mathrm{e}^{-b(t,r)}}{r}\,\vartheta^\theta\,, \qquad \omega^\phi{}_t = 0\,, \qquad \omega^\theta{}_t = 0\,, \tag{8}$$

where a prime and superscript dot denote derivatives with respect to $r$ and $t$, respectively.

*Remark:* Since for any metric-compatible connection (like the Levi–Civita connection studied here) we have $\mathrm{d}g_{ij} = 0$ and therefore $\omega_{ij} = -\,\omega_{ji}$, the connection only has six independent components in four dimensions, while the Christoffel symbols have 40 independent components.

**5.3** Calculate the curvature 2-forms $\Omega^i{}_j$ by using the second Cartan structure equation

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_a \wedge \omega^a{}_j =: \frac{1}{2}\,R^i{}_{jkl}\,\vartheta^k \wedge \vartheta^l \tag{9}$$

and read off the anholonomic components $R^i{}_{jkl}$ of the Riemann curvature tensor.

*See reverse.*

*Intermediate result:* The non-vanishing anholonomic components of the Riemann curvature tensor read

$$R^t{}_{rtr} = -\left[\mathrm{e}^{-2b}(a'^2 - a'b' + a'') + \mathrm{e}^{-2a}(\dot{a}\dot{b} - \dot{b}^2 - \ddot{b})\right] = -R^r{}_{trt}\,, \tag{10}$$

$$R^t{}_{\theta t\theta} = R^t{}_{\phi t\phi} = -\frac{a'\,\mathrm{e}^{-2b}}{r}\,, \qquad R^r{}_{\theta r\theta} = R^r{}_{\phi r\phi} = \frac{b'\,\mathrm{e}^{-2b}}{r}\,, \tag{11}$$

$$R^t{}_{\theta r\theta} = R^t{}_{\phi r\phi} = -\frac{\dot{b}\,\mathrm{e}^{-a-b}}{r} = -R^r{}_{\theta t\theta} = -R^r{}_{\phi t\phi}\,, \qquad R^\phi{}_{\theta\phi\theta} = \frac{1-\mathrm{e}^{-2b}}{r^2}\,. \tag{12}$$

**5.4** Determine the anholonomic components of the Ricci tensor $R_{ij} = R^a{}_{iaj}$ as well as the Ricci scalar $R = \eta^{ij}R_{ij}$. Note that for the contraction of anholonomic indices the Minkowski metric has to be used.

**5.5** For a diagonal metric the holonomic components of a $(1,1)$-tensor coincide with the anholonomic components.*

Therefore calculate the mixed components of the Einstein tensor

$$G^i_{\ j} \stackrel{*}{=} G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\,\delta^\mu_\nu\,R\,. \tag{13}$$

**5.6** Back to physics. Outside of the mass distribution we have a vacuum and therefore $T^\mu{}_\nu = 0$, i. e. $G^\mu{}_\nu = 0$. Use this to show that $b$ depends only on $r$ and that $a(r) = -b(r)$.

**5.7** Finally, integrate the differential equation arising from $G^0{}_0 = 0$ to find a relation between $b$ and $r$ and use the Newtonian limit to fix the integration constant. Show that

$$\mathrm{e}^{-2b(r)} = 1 - \frac{2GM}{r}\,, \tag{14}$$

which concludes our derivation of the Schwarzschild solution.

*Note that this consideration is absolutely crucial given that the Einstein equation is formulated for coordinate indices and not for anholonomic indices.