University of Cologne http://www.thp.uni-koeln.de/gravitation/courses/rcii18.html
Institute for Theoretical Physics
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2nd exercise sheet on Relativity and Cosmology II

Summer term 2018

Deadline for delivery: Thursday, 26th April 2018 during the exercise class.

Exercise 33 (20 points): *Effective Schwarzschild potential*

The aim of this exercise is to analyze certain properties of the movement of massive test particles in the Schwarzschild spacetime.
For this purpose, consider the equation of motion on the equatorial plane $\vartheta = \pi/2$ with an effective potential V_{eff} that results from the geodesic equation

$$\frac{\dot{r}^2}{2} + V_{\text{eff}}(r) = E\,, \qquad V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}.$$

Here ℓ and E indicate constants of motion.

In the following, express radial distances in terms of the Schwarzschild radius $r_S = 2GM$.

33.1 **a)** Which conditions does a test particle approaching from infinity ($r \to +\infty$) have to fulfill in order to fall into the center of the effective potential?

b) Under which circumstances does a particle that starts from rest at infinity fall into the center?

Compare your results to the situation in Newtonian gravity.

33.2 **a)** In which cases do bound particle orbits exist?

b) For which values of the radial coordinate r are there stable orbits?

c) Under which conditions are there instable orbits and what happens to a particle which is slightly deflected from such an orbit?

d) Sketch the potential V_{eff} qualitatively for the cases $\ell/GM < 2\sqrt{3}$, $\ell/GM = 2\sqrt{3}$ and $\ell/GM > 2\sqrt{3}$ and show the following statements:

i. For $\ell/GM < 2\sqrt{3}$ every infalling particle falls towards the event horizon $r = 2GM$.

ii. The most strongly bound stable orbit is located at $r = 6GM$ with $\ell/GM = 2\sqrt{3}$ and it possesses a relative binding energy of $1 - \sqrt{8/9}$.

33.3 Consider a massive test particle initially being at rest at the radial coordinate $R > r_S$ that falls radially ($\ell = 0$) into the center.

a) Find the solution of the resulting initial value problem.
*H*int: The solution can be given in a parametrized form $r(\eta)$, $\tau(\eta)$ (where τ is the proper time of the particle) which describes a *c*ycloid orbit.
At which proper time τ_0 does the particle reach the center of potential?

b) How long does it take for this particle to reach the Schwarzschild radius as measured by an observer at infinity?

33.4 In the lecture it was mentioned that Kepler's Third law holds for circular orbits in the form

$$GM = \omega^2 r^3\,,$$

where $\omega = d\phi/dt$. Prove this.