University of Cologne http://www.thp.uni-koeln.de/gravitation/courses/rcii18.html
Institute for Theoretical Physics
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3rd exercise sheet on Relativity and Cosmology II

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Deadline for delivery: Thursday, 3rd May 2018 during the exercise class.

Exercise 34 (15 credit points): *Differential forms*

34.1 Consider an n-dimensional manifold with a metric. Let $\{\omega^i\}$ be an orthonormal co-basis of 1-forms, and let ω be the preferred volume form $\omega = \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^n$.

Show that, in an arbitrary coordinate system $\{x^k\}$, the following holds:

$$\omega = \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \,, \tag{1}$$

where g denotes the determinant of the metric, whose components g_{ij} are given in these coordinates.

34.2 The contraction of a p-form ω (with components $\omega_{ij\ldots k}$) with a vector v (with components v^i) is given by $[\omega(v)]_{j\ldots k} = \omega_{ij\ldots k}\, v^i$. Consider the n-form $\omega = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$.

Show that, with a given vector field v, the following holds:

$$d[\omega(v)] = v^i{}_{,i}\, \omega \,. \tag{2}$$

34.3 Define $(\mathrm{div}_\omega v)\, \omega \coloneqq d[\omega(v)]$.

Show that, by using coordinates in which ω has the form $\omega = f\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, the following holds:

$$\mathrm{div}_\omega v = \frac{1}{f}\left(f v^i\right)_{,i} \,. \tag{3}$$

34.4 In three-dimensional Euclidean space, the preferred volume form is given by $\omega = dx \wedge dy \wedge dz$.

Show that, in spherical coordinates, this volume form is given by $\omega = r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi$.
Use the result of **34.3** to show that the divergence of a vector field

$$v = v^r \frac{\partial}{\partial r} + v^\theta \frac{\partial}{\partial \theta} + v^\phi \frac{\partial}{\partial \phi} \tag{4}$$

is given by

$$\mathrm{div}\, v = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 v^r\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta\, v^\theta\right) + \frac{\partial v^\phi}{\partial \phi} \,. \tag{5}$$

Exercise 35 (5 credit points + *8 bonus points*): *Electrodynamics in flat spacetime*

Differential forms are a convenient tool for field theories, as we will show in this exercise on the example of Maxwell electrodynamics in Minkowski spacetime. We know that electromagnetic field strength is given by Faraday antisymmetric tensor $F_{\mu\nu}$, and the current is given by j^μ. In the language of exterior calculus, $F_{\mu\nu}$ are components of the Faraday 2-form $\mathbf{F}$ describing an arbitrary electromagnetic field, given by

$$\mathbf{F} \coloneqq \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = -E_x\, dt \wedge dx - E_y\, dt \wedge dy - E_z\, dt \wedge dz + B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy \,, \tag{6}$$

while j^μ are components of the current 1-form $\mathbf{j}$ is given by

$$\mathbf{j} \coloneqq j_\mu\, dx^\mu = -\rho\, dt + j_x\, dx + j_y\, dy + j_z\, dz \,. \tag{7}$$

See reverse.

35.1 The Hodge star operator $\star$ maps p-forms to $(4-p)$-forms. Therefore, 2-forms are mapped to 2-forms by this operator. The 2-form dual to the Faraday 2-form is the Maxwell 2-form, defined by $\mathbf{G} := \star\mathbf{F}$.

The Hodge star operator acts on the 2-form basis as follows

$$\star(dx^\mu \wedge dx^\nu) = \frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\varepsilon_{\alpha\beta\gamma\delta}dx^\gamma \wedge dx^\delta\,, \tag{8}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the inverse Minkowski metric and $\varepsilon_{\alpha\beta\gamma\delta}$ totally anti-symmetric Levi-Civita tensor density ($\varepsilon_{0123} = +1$). For example,

$$\star(dt \wedge dx) = -dy \wedge dz\,, \quad \text{etc.} \tag{9}$$

Show that the Maxwell 2-form is given by

$$\mathbf{G} = B_x\, dt \wedge dx + B_y\, dt \wedge dy + B_z\, dt \wedge dz + E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy\,. \tag{10}$$

Which relation between the components of **F** and **G** holds?

35.2 ***Bonus exercise.*** With these definitions, the Maxwell equations can be written in a compact form: $d\mathbf{F} = 0$ and $d\mathbf{G} = 4\pi \star \mathbf{j}$, therefore,

a) Show that the equation $d\mathbf{F} = 0$ corresponds to the two homogeneous Maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0\,. \tag{11}$$

b) Calculate $\star\mathbf{j}$ (a 3-form), which is the dual of the current 1-form **j**. For that, use the following relations

$$\star dx^\mu = \frac{1}{3!}\eta^{\mu\alpha}\varepsilon_{\alpha\beta\gamma\delta}\, dx^\beta \wedge dx^\gamma \wedge dx^\delta\,, \tag{12}$$

for example

$$\star dt = -dx \wedge dy \wedge dz\,, \quad \text{etc.} \tag{13}$$

c) Then show that the equation $d\mathbf{G} = 4\pi \star \mathbf{j}$ corresponds to the two inhomogeneous Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad \text{and} \quad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 4\pi\vec{j}\,. \tag{14}$$

d) What can you do to turn eq. (11) into eq. (14) in the vacuum case?

35.3 ***Bonus exercise.*** The exterior derivative is nilpotent, i.e. the relation $d(d\boldsymbol{\omega}) = 0$ holds for any p-form $\boldsymbol{\omega}$. Choosing $\boldsymbol{\omega} = \star\mathbf{j}$, show that this yields the continuity equation $\partial_t \rho - \vec{\nabla} \cdot \vec{j} = 0$.

35.4 ***Bonus exercise.*** Explain in your own words why a covariant derivative is needed on a curved background, and give special attention to the connection. What is the intuitive interpretation of the connection? Why can it be chosen to be zero in Minkowski space?