Skewonless media with no birefringence.

A general constitutive relation.

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- Impact: Propels new findings, confirms past ones + Fronts in non-linear media + Inserting skewon.

Linear local constitutive relation (pre-metric).

The EM fields **D** and **H** are related to **E** and **B** linearly

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$$W^I = \chi^{IJ} F_J \ \Rightarrow \ W^{\mu\nu} = \chi^{\mu\nu J} F_J \ \Rightarrow \ W^{\mu\nu} = \frac{1}{2}\chi^{\mu\nu\alpha\beta} F_{\alpha\beta} \ .$$

Relate $W^{\mu\nu}$, $F_{\alpha\beta}$ with respective 6 independent entries:

$$[F_{\alpha\beta}] = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix} ,$$

$$[W^{\mu\nu}] = \begin{bmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{bmatrix} ,$$

Important symmetry:

$$\chi^{\mu\nu\alpha\beta} = -\chi^{\nu\mu\alpha\beta} = -\chi^{\mu\nu\beta\alpha},$$

(E.J. Post, "Formal Structure of Electromagnetics", 1962).

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Finite skewon violates usual $\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}^{\,T}$, $\bar{\bar{\mu}} = \bar{\bar{\mu}}^{\,T}$, $\bar{\bar{\alpha}} = -\bar{\bar{\beta}}^{\,T}$.

Axion ${}^{[3]}\chi^{\mu\nu\alpha\beta}$ is rare but possible.

Axion obeys ${}^{[3]}\chi^{\mu\nu\alpha\beta} \propto e^{\mu\nu\alpha\beta} \in \{\pm 1, 0\}$. In nature:

- Static: de Lange & Raab, J. Opt. A, 2001.
- Waves: Hehl, Obukhov, Rivera & Schmid, PLA, 2008.

Given these considerations the skewon is assumed to vanish.

Dispersion (Fresnel) relation.

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- Quartic in waves' $K_\mu = (-\omega, k_i)$, cubic in media's $\chi^{\alpha\beta\mu\nu}$:

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- Example: vacuum, simplest non-birefringent medium $\chi_0^{\mu\nu\alpha\beta} = \sqrt{-\det(g_{\alpha\beta})}(\mu_0/\varepsilon_0)^{-\frac{1}{2}} \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}\right)$.

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- Fix: adapted from Schuller. 4D transform 6×6 dets:

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- Factors in red can compensate each other. Reconcile:

$$\frac{\chi^{\mu\nu\alpha\beta}}{|\det(\chi^{IJ})|^{1/6}} = \frac{\Omega^{\mu\nu\alpha\beta}}{|\det(\Omega^{IJ})|^{1/6}} \ ,$$

matches pre-metric well; but how about axiomatics?

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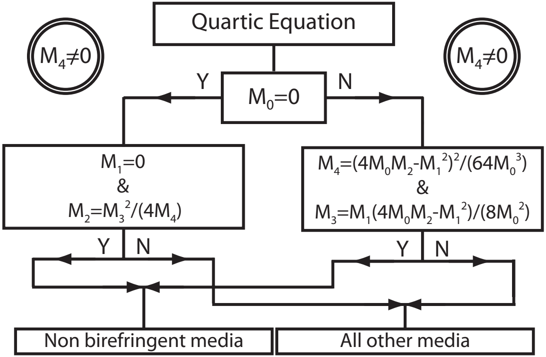

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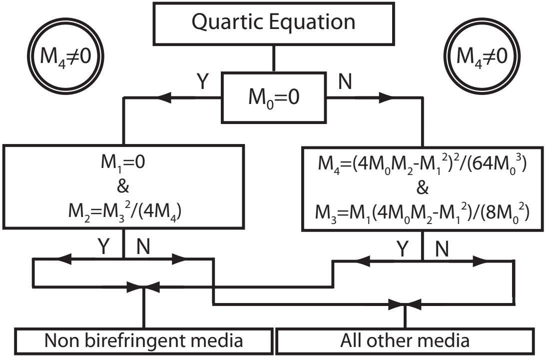

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- Non-birefringent = optical metric + (bivectors, axion).

Mathematica®: 23 matrices in, 5 matrices out.

Lämmerzahl/Hehl/Itin birefringence elimination gives 5 χ^{IJ}:

$$\begin{bmatrix} -\tau & 0 & 0 & \sigma & 0 & 0 \\ 0 & -\tau & 0 & 0 & \sigma & 0 \\ 0 & 0 & -\tau & 0 & 0 & \sigma \\ \sigma & 0 & 0 & \tau & 0 & 0 \\ 0 & \sigma & 0 & 0 & \tau & 0 \\ 0 & 0 & \sigma & 0 & 0 & \tau \end{bmatrix} \qquad \begin{bmatrix} \lambda_5 & 0 & 0 & \lambda_6 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & \lambda_2 \\ \lambda_6 & 0 & 0 & \lambda_5 & 0 & 0 \\ 0 & \lambda_4 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & \lambda_1 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & \pm(\lambda_3-\lambda_5) & 0 & 0 & \lambda_3 \\ \lambda_3 & 0 & 0 & \epsilon_1 & 0 & 0 \\ 0 & \lambda_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & \pm(\lambda_3-\lambda_5) \end{bmatrix}$$

Mathematica®: 23 matrices in, 5 matrices out.

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- Magneto-electric diagonals $\rightarrow \chi^{\mu\nu\alpha\beta}$ has axion $\alpha e^{\mu\nu\alpha\beta}$.

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- Magneto-electric diagonals $\rightarrow \chi^{\mu\nu\alpha\beta}$ has axion $\alpha e^{\mu\nu\alpha\beta}$.
- "Rectangles" $\rightarrow \chi^{\mu\nu\alpha\beta}$ has bivector terms $A^{\mu\nu}A^{\alpha\beta}$.

"Hint": analytic $\chi^{\mu\nu\alpha\beta}$ suggested by 5 matrices.

$$\chi^{\mu\nu\rho\sigma} = |\det(G^{-1}_{\alpha\beta})|^{\frac{1}{2}} M\big[\,(G^{\mu\rho}G^{\nu\sigma} - G^{\mu\sigma}G^{\nu\rho}) + s_A A^{\mu\nu}A^{\rho\sigma} + s_{\tilde{A}}\tilde{A}^{\mu\nu}\tilde{A}^{\rho\sigma}\big] + \alpha e^{\mu\nu\rho\sigma}$$

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Too coarse, skewonless: nonbirefringent + some birefringent.

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KILL!

Refine analytic law, get necessary & sufficient.

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Substitute analytic "hint" into dispersion relation $f(\mathbf{K})$:

$$\left[(\overline{\overline{G}} - s_A \mathbf{A} \cdot \overline{\overline{G}}^{-1} \cdot \mathbf{A})^{\mu\nu} K_\mu K_\nu\right]\left[(\overline{\overline{G}} - s_{\tilde{A}} \tilde{\mathbf{A}} \cdot \overline{\overline{G}}^{-1} \cdot \tilde{\mathbf{A}})^{\rho\sigma} K_\rho K_\sigma\right]$$
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Birefringent. Become non-birefringent $(G^{\alpha\beta} K_\alpha K_\beta)^2 = 0$ if:

$$\mathbf{A} \cdot \overline{\overline{G}}^{-1} \cdot \mathbf{A} \cdot \overline{\overline{G}}^{-1} = a_1 \mathbb{1} \ , \quad \tilde{\mathbf{A}} \cdot \overline{\overline{G}}^{-1} \cdot \tilde{\mathbf{A}} \cdot \overline{\overline{G}}^{-1} = a_2 \mathbb{1} \ ,$$
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Density $e^{\mu\nu\alpha\beta}$ converted to tensor $\epsilon_G^{\mu\nu\alpha\beta}$ via optical $G^{\alpha\beta}$. Also:

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Results: all skewonless non-birefringent $\chi^{\mu\nu\alpha\beta}$.

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When the signature of the optical $G^{\alpha\beta}$ is Lorentzian (3,1).

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Other signatures: only metamaterials, hyperlens (Jacob '06).

Ties together previous literature.

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1. EM reciprocity: axionless (tilde) part obeys closure

$$\hat{e}_{IK}\tilde{\chi}^{KL}\hat{e}_{LM}\tilde{\chi}^{MJ} = -\lambda^2\delta_I^J.$$

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(3,1): Hodge + axion $\Rightarrow$ $(G^{\alpha\beta}K_\alpha K_\beta)=0$. CONVERSE?

"Note that the vanishing of birefringence is not equivalent to the validity of the reciprocity [closure] relation." (L&H, '04)

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Quantise skewonless bi-anisotropic $W^I = \chi^{IJ} F_J$. Study general Casimir effect. Usual Casimir = no birefringence.

Matias Dahl (arXiv:1103.3118).

Confirms the result "(3,1): Hodge + axion $\Rightarrow$ $(G^{\alpha\beta} K_\alpha K_\beta) = 0$. CONVERSE!" using Gröbner bases.

Conclusions.

A. Favaro, L. Bergamin, “The non-birefringent limit of all linear, skewonless media and its unique light-cone structure.”, Annalen der Physik, 1-19 (2011).

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- Showed how nicely it all fits in the literature.

Beyond: Non-Linear or with Skewon.

Obukhov & Rubilar (PRD, '02).

- Discuss very general non-linear Lagrangian (Skewon=0):

$$L = L(I_1, I_2)\ , \qquad I_1 = F_{\mu\nu}F^{\mu\nu}\ , \qquad I_2 = F_{\mu\nu}\tilde{F}^{\mu\nu}\ ,$$

- Return to linear optics if propagating very sharp front.
- For this skewonless medium, no-birefringence if only if

$$\chi_{OR}^{\mu\nu\rho\sigma} = |\det(G^{-1}_{\alpha\beta})|^{\frac{1}{2}} M \left[G^{\mu\rho}G^{\nu\sigma} - G^{\mu\sigma}G^{\nu\rho} \right] + \alpha e^{\mu\nu\rho\sigma}$$

as per this talk. (Made this statement more explicit).

Skewon: divergence from Hodge dual (Bergamin, '10).

$\chi^{\mu\nu\alpha\beta} \propto A^{\mu\nu}A^{\alpha\beta} + {}^{[2]}\chi^{\mu\nu\alpha\beta}$ is non-birefringent.

Thank-you!