Covariant Methodology in Transformational Acoustics

A. Favaro[1] M.W. McCall[1] P. Kinsler[1] L. Bergamin[2]
S. Carloni[3]

[1]Imperial College of Science, Technology and Medicine
London, UK

[2]KBP GmbH
Bern, CH and [3]Advanced Concepts Team
ESTEC

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Outline

1. Geometric (‘Ray’) Transformation Acoustics
   - Possible approaches to transformation acoustics
   - Parallel transport and geodesics
   - Example: acoustic ray cloak
2. Linear elastodynamics
3. Equations of motion for linearised elastodynamics
4. Transformation acoustics for special materials
   - Intertial transformation acoustics
   - Pentamode transformation acoustics
5. Example: cylindrical acoustic cloak
6. TA beyond pentamodes
7. Conclusions

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Transformation Acoustic

Possible Approaches

- Spatial e-m cloak is the canonical e-m example
- The transformation electromagnetics programme is (in principle) exact, embracing, for example the near field.
- Analogue transformation theories in (e.g.) acoustics appear to be mathematically harder to achieve
- First seek a geometrical *ray* based theory - not exact, but potentially very general

Field

$(\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H})$

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$(\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H})$

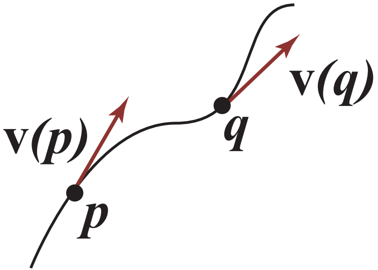
Physical, ϕ

Scalar wave equation

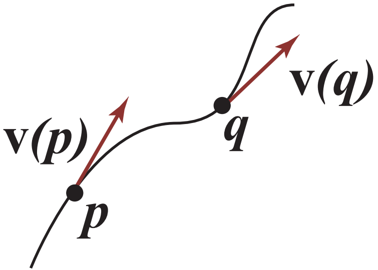
Geometrical, $\nabla\phi$

Eikonal equation

Parallel transport and geodesics



Parallel transport and geodesics



Covariant Derivative

A covariant derivative provides a means of comparing vectors at different locations

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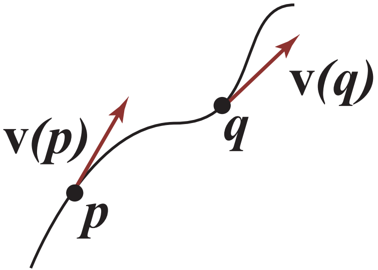
Parallel transport and geodesics



Geodesics

Parallel transporting **v** along itself generates a *geodesic*

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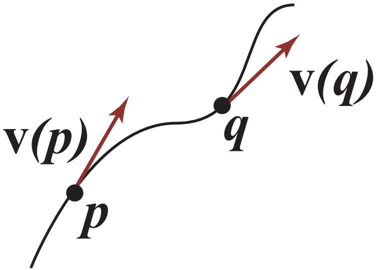
Parallel transport and geodesics



Geodesic Equation

$$\mathbf{v} = \frac{dx^i}{dt}\partial_i \Rightarrow \frac{d^2x^i}{dt^2} + \Gamma^j{}_{ki}\frac{dx^k}{dt}\frac{dx^i}{dt} = 0, \textit{ connection coefficients}, \Gamma^j{}_{ki}$$

Metric and Covariant Derivative

- Given metric g_{ij}
- Natural covariant derivative from condition $D_k g_{ij} = 0$:

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{im} \left(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk} \right)$$

e.g. Euclidean metric in Cartesians $\Rightarrow g_{ij} = \delta_{ij}$ and $\Gamma^i{}_{jk} = 0$

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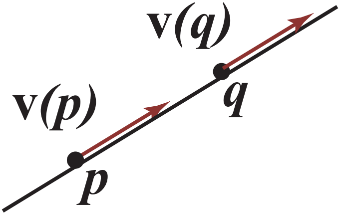
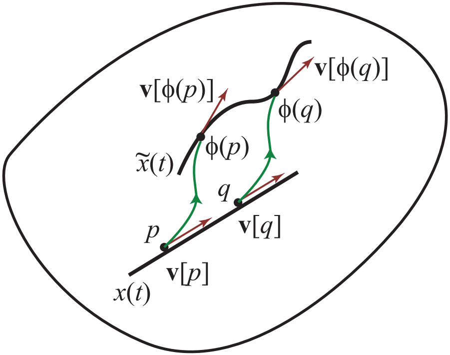
Straight lines



Cartesians with $\Gamma^{i}{}_{jk} = 0$

$$\frac{d^2 x^i}{dt^2} = 0 \Rightarrow x^i(t) = x^i(0) + v_0^i t$$

Diffeomorphism $\varphi : p \to \varphi(p)$ defines a *new* covariant derivative



Demanding new curve $\tilde{x}(t)$ is a *new* geodesic requires

$$\bar{\Gamma}^{l}{}_{mn} = (\varphi_*)^{l}{}_{i}\,(\varphi^*)^{j}{}_{m}\,(\varphi^*)^{k}{}_{n}\,\Gamma^{i}{}_{jk} + (\varphi_*)^{l}{}_{j,k}\,(\varphi^*)^{j}{}_{m}\,(\varphi^*)^{k}{}_{n}\ ,$$

where $(\varphi^*)^{i}{}_{l} = \partial\tilde{x}^{i}/\partial x^{l}$, and $(\varphi^*)^{i}{}_{l}\,(\varphi_*)^{l}{}_{j} = \delta^{i}{}_{j}$

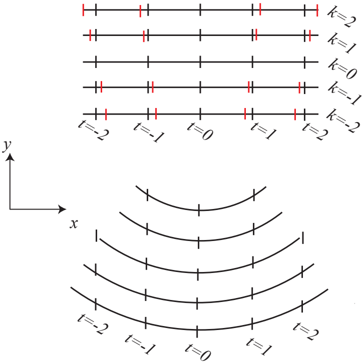
Conformal transformation (aka spatial dilation) defines a *new* covariant derivative

Dilate space $ds \to n ds$

- New metric (in Cartesians):

$$\bar{g} = n^2 \delta_{ij}$$

- New covariant derivative, $\bar{D}_i$



New connection coefficients (in Cartesians)

$$\bar{\Gamma}^i{}_{jk} = \frac{1}{2}\delta^{im}\left(\delta_{mj}\partial_k + \delta_{mk}\partial_i - \delta_{jk}\partial_m\right)[\ln n]$$

Demand equality between two new covariant derivatives

Morphed geodesics φ = geodesics generated by $g_{ij} = n^2\delta_{ij}$

$$(\phi_*)^l{}_i\,(\phi^*)^j{}_m\,(\phi^*)^k{}_n\,\Gamma^i{}_{jk} - (\phi^*)^l{}_{j,k}\,(\phi^*)^j{}_m\,(\phi^*)^k{}_n =$$
$$\delta^i{}_j\,(\ln n)_{,k} + \delta^i{}_k\,(\ln n)_{,j} - \delta^{im}\delta_{jk}\,(\ln n)_{,m} \ .$$

Key result, linking some desired deformation $\varphi : p \to \varphi(p)$ of the (linear) geodesics of a homogeneous medium to the index distribution (i.e. $n(p)$) of an inhomogeneous medium required to achieve that deformation.

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Example: Acoustic Ray Cloak

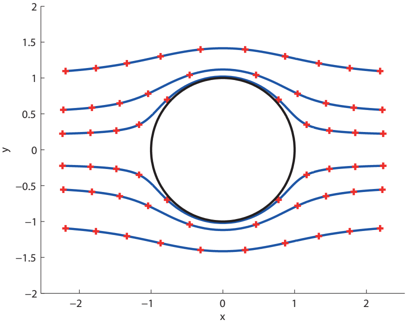
Diffeomorphism, φ

$$\varphi : [r, \theta] \to \left[\left(r^2 + a^2 \right)^{1/2}, \theta \right]$$

Running this through our algorithm yields

$$n = \left(\frac{\rho}{\lambda}\right)^{1/2} = \left(\frac{\rho_0}{\lambda_0}\right)^{1/2} \left(1 - \frac{a^2}{r^2}\right)^{1/2}$$

Mass density, ρ
Bulk modulus, λ



Scope - Towards a general theory

Physics	Field(s)	Field Equation(s)	Medium Parameter(s)
Maxwell	F, H	$F_{\alpha\beta,\gamma} = 0$, $G^{\alpha\beta}{}_{,\beta} = 0$	Constitutive tensor $\chi^{\alpha\beta\mu\nu}$
Acoustics	p, ρ	$\nabla \cdot (\rho \mathbf{v}) + \partial_t \rho = 0$ $-\nabla p = \rho \left[\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}\right]$	Bulk modulus B Unperturbed density, ρ_0 Unperturbed velocity, $\mathbf{v}_0$
Diffusion Heat	ρ T	$\partial_t \rho = D \nabla^2 \rho$ $\partial_t T = K \nabla^2 T$	Diffusion constant, D Thermal conductivity, K
Schrodinger	ψ	$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi$	Potential, $V(t, \mathbf{r})$

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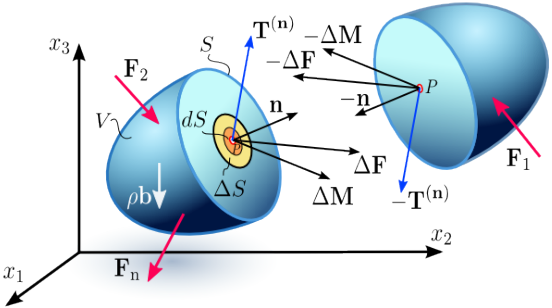
The stress tensor



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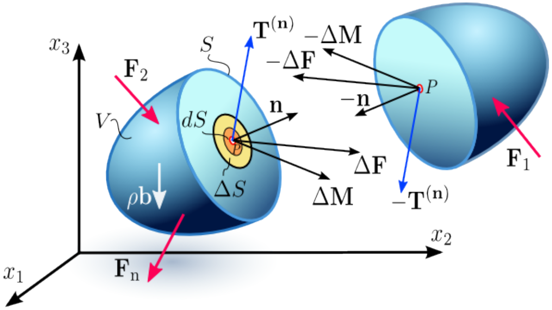
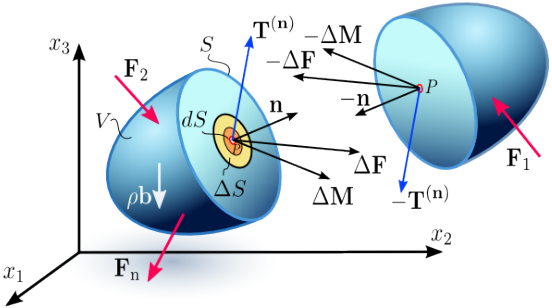


Stress vector

$$T^{(\boldsymbol{n}),i} = \frac{dF^i}{dS}$$

Force per area on (virtual) surface in the body.

The stress tensor



Stress tensor

$$T^{(\boldsymbol{n})\,i} = n^j \sigma_j{}^i \qquad \text{Cauchy's stress theorem}$$

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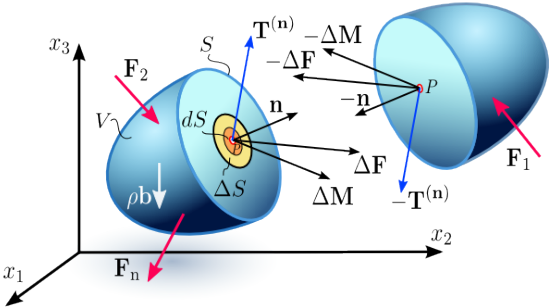
The stress tensor



Momentum balance

$$D_k \sigma_i{}^k + \rho g_{ik} F^k = \rho g_{ik} W^k \qquad \rho \boldsymbol{F} \text{ volumic forces, } \boldsymbol{W} = \ddot{\boldsymbol{u}} \text{ acceleration.}$$

$$g^{ik} \sigma_k{}^j = g^{jk} \sigma_k{}^i \qquad \sigma^{ij} \text{ symmetric, angular momentum balance.}$$

Strain tensor

Strain tensor: deformation of body due to displacement field $\boldsymbol{u}$.
Infinitesimal displacement:

$$\boldsymbol{e} = \mathcal{L}_{\boldsymbol{u}}\boldsymbol{g} \qquad \boldsymbol{g} \text{ metric, } \mathcal{L}_{\boldsymbol{u}} \text{ Lie derivative w.r.t. } \boldsymbol{u}$$

In components

$$e_{ij} = \frac{1}{2}\Big(D_i(g_{jk}u^k) + D_j(g_{ik}u^k)\Big) = \frac{1}{2}(D_i u_j + D_j u_i)$$

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Linear stress-strain relation

Elasticity tensor $\boldsymbol{C}$

$$\sigma_i{}^j = C_i{}^{jkl} e_{kl}$$

subject to constraints

$$C^{ijkl} = C^{jikl} = C^{ijlk}$$

$C^{ijkl} = C^{klij}$ if elastic potential exists ("skewon free").

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General linear elastodynamics

Recall:

1. Momentum balance for $\boldsymbol{\sigma}$, no volumic forces

$$D_k {\sigma_i}^k = \rho g_{ik} \ddot{u}^k$$

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Use 3 in 1

$$D_k\left(C_i^{\,kmn} e_{mn}\right) = \rho g_{ik} \ddot{u}^k$$

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Use 2 in 3 in 1

$$D_k\left(C_i^{\,kmn} D_m(g_{nj}u^j)\right) = \rho g_{ik} \ddot{u}^k$$

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$$\sigma_i^j = C_i^{jkl} e_{kl}$$

Use 2 in 3 in 1 with metric compatibility

$$D_k\left({C^{ikm}}_j D_m u^j\right) = \rho \ddot{u}^i$$

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General linear elastodynamics

$$D_k\left(C^{ikm}{}_j D_m u^j\right) = \rho \ddot{u}^i$$

- is a second order DEQ for $\boldsymbol{u}$ (wave equation),

General linear elastodynamics

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Simple TA scheme of full linear elastodynamics not possible.

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Anisotropic mass

Replace the mass density ρ by tensor ${\rho^i}_j$

$$D_k {\sigma_i}^k = \rho g_{ik} \ddot{u}^k$$

$$\Downarrow$$

$$D_k {\sigma_i}^k = g_{ij} {\rho^j}_k \ddot{u}^k$$

and

$$D_k \left({C_i}^{kmn} D_m (g_{nj} u^j) \right) = \rho g_{ik} \ddot{u}^k$$

$$\Downarrow$$

$$D_k \left({C_i}^{kmn} D_m (g_{nj} u^j) \right) = g_{ij} {\rho^j}_k \ddot{u}^k$$

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Eigentensor decomposition

For any skewon free material there exists a decomposition

$$C_i^{\ jkl} = g_{im} \sum_{J=1}^{6} \lambda_J S_J^{mj} S_J^{kl}$$

with λ_J eigenvalues, $S_J^{ij} = S_J^{ji}$ eigentensors.

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Background: Map double indices (ij) to single index α

$$\alpha = (1, 2, 3, 4, 5, 6) = ((11), (22), (33), (23), (13), (12))$$

then

$$C^{ijkl} \Longrightarrow C^{\alpha\beta} = C^{\beta\alpha} \qquad \text{skewon free}$$

C can be mapped on a real, symmetric 6×6 matrix.

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Pentamode materials

Only one eigenvalue $\lambda \neq 0$:

$$C_i^{jkl} = g_{im} \lambda S^{mj} S^{kl}$$

- Five “easy” modes
- Wave equation in one mode, “pseudo-pressure”

$$\tilde{p} = -\lambda S^{ij} e_{ij} = -\lambda \xi$$

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Anisotropic fluid

Pentamode with condition $\boldsymbol{S} \propto \boldsymbol{g}$.

$$C_i{}^{jkl} = g_{im} \lambda g^{mj} g^{kl} = \lambda \delta_i^j g^{kl}$$

Only $p = -\lambda \xi$ couples
Notice

$$T^{(n)\,i} = n^j \sigma_j{}^i = n^i \lambda (g^{kl} e_{kl}) = n^i \lambda \xi$$

Inertial transformation acoustics

Momentum balance

$$D_k \sigma_i{}^k = g_{ij} \rho^j{}_k \ddot{u}^k$$

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Inertial transformation acoustics

Momentum balance with constutive law

$$D_k \left(C_i^{\ kmn} e_{mn} \right) = g_{ij} \rho^j{}_k \ddot{u}^k$$

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Inertial transformation acoustics

Momentum balance with constutive law for anisotropic fluid

$$D_k \left(\lambda \delta_i^k g^{mn} e_{mn} \right) = g_{ij} \rho^j{}_k \ddot{u}^k$$

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Inertial transformation acoustics

Momentum balance with constutive law for anisotropic fluid

$$D_i\left(\lambda\xi\right) = g_{ij}\rho^j{}_k\ddot{u}^k \qquad \xi = g^{mn}e_{mn}$$

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Strain tensor

$$e_{ij} = \frac{1}{2}\Big(D_i(g_{jk}u^k) + D_j(g_{ik}u^k)\Big)$$

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$\not{e}$ "easy modes", drops out

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Inertial transformation acoustics

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Strain tensor

$$e_{ij} = g_{im}\not{e}^m{}_j + \left(D_k u^k\right)g_{ij}$$

$\not{e}$ "easy modes", drops out

Wave equation in first order for anisotropic fluid

$$D_i\left(\lambda\xi\right) = g_{ij}\rho^j{}_k\ddot{u}^k$$
$$\xi = D_i u^i$$

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Inertial transformation acoustics

Wave equation in first order for anisotropic fluid $g = \det(g_{ij})$

$$D_i\left(\lambda\xi\right) = \partial_i\left(\lambda\xi\right) = g_{ij}\rho^j{}_k\ddot{u}^k$$

$$\xi = D_i u^i = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}u^i\right)$$

- DEQ transformed to scalar wave equation

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Inertial transformation acoustics

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Inertial transformation acoustics

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- DEQ transformed to scalar wave equation
- All covariant derivatives transformed to plain derivatives
- Equations not covariant! ⇔ Electrodynamics

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Inertial transformation acoustics theorem

If ξ and $\boldsymbol{u}$ are a solution of

$$\partial_i(\lambda\xi) = g_{ij}\rho^j{}_k\ddot{u}^k \qquad \xi = D_i u^i = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}u^i\right)$$

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Inertial transformation acoustics theorem

If ξ and $\bar{\boldsymbol{u}}$ after transformation $\boldsymbol{x} \to \bar{\boldsymbol{x}}(x)$ are a solution of

$$\bar{\partial}_i(\lambda\xi) = \bar{g}_{ij}\bar{\rho}^j{}_k\ddot{\bar{u}}^k \qquad \xi = \bar{D}_i\bar{u}^i = \frac{1}{\sqrt{\bar{g}}}\bar{\partial}_i\left(\sqrt{\bar{g}}\bar{u}^i\right)$$

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Inertial transformation acoustics theorem

If ξ and $\bar{\boldsymbol{u}}$ after transformation $\boldsymbol{x} \to \bar{\boldsymbol{x}}(x)$ are a solution of

$$\bar{\partial}_i(\lambda\xi) = \bar{g}_{ij}\bar{\rho}^j{}_k\ddot{\bar{u}}^k \qquad \xi = \bar{D}_i\bar{u}^i = \frac{1}{\sqrt{\bar{g}}}\bar{\partial}_i\left(\sqrt{\bar{g}}\bar{u}^i\right)$$

then

$$\xi' = \frac{\sqrt{\bar{g}}}{\sqrt{g'}}\xi \qquad u'^i = \frac{\sqrt{\bar{g}}}{\sqrt{g'}}\bar{u}^i$$

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transformation acoustics = transformation + reinterpretation

Pentamode transformation acoustics

- $g^{ij} \Rightarrow S^{ij}$ in material parameters

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- Stress DEQ

$$\partial_i(\lambda\xi) = S^{-1}_{ij} \rho^j{}_k g^{kl} S^{-1}_{lm} \ddot{w}^m \qquad \text{or} \qquad S^{ml} g_{lk} \rho^{-1\,k}{}_j S^{ji} \partial_i(\lambda\xi) = \ddot{w}^m$$

Either $\boldsymbol{S}$ or ρ have to be invertible!

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Pentamode transformation acoustics theorem

If ξ and $\boldsymbol{w}$ are a solution of

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$$D'_i S'^{ij} = 0 \qquad \lambda' = \frac{\sqrt{g'}}{\sqrt{\bar{g}}}\lambda'$$

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Reinterpretation in pentamode TA

$$\lambda' = \frac{\sqrt{g'}}{\sqrt{\bar{g}}}\lambda'$$

$$S'^{-1}_{ij}\rho'^{j}{}_{k}g'^{kl}S'^{-1}_{lm} = \frac{\sqrt{g'}}{\sqrt{\bar{g}}}\bar{S}^{-1}_{ij}\bar{\rho}^{j}{}_{k}\bar{g}^{kl}\bar{S}^{-1}_{lm} \qquad \text{or its inverse}$$

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1. transformation acoustics = transformation + reinterpretation

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1. transformation acoustics = transformation + reinterpretation
2. Reinterpretation of modulus analogous to inertial TA

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1. transformation acoustics = transformation + reinterpretation
2. Reinterpretation of modulus analogous to inertial TA
3. Reinterpretation of $\boldsymbol{S}$ and ρ not unique
4. Reinterpretation includes DEQ, not algebraic!

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Outline

1. Geometric ('Ray') Transformation Acoustics
   - Possible approaches to transformation acoustics
   - Parallel transport and geodesics
   - Example: acoustic ray cloak
2. Linear elastodynamics
3. Equations of motion for linearised elastodynamics
4. Transformation acoustics for special materials
   - Intertial transformation acoustics
   - Pentamode transformation acoustics
5. Example: cylindrical acoustic cloak
6. TA beyond pentamodes
7. Conclusions

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Cylindrical acoustic cloak

Original medium

Simple fluid with constant bulk modulus

$$C_i^{jkl} = \lambda \delta_i^j g^{kl} \qquad \lambda = \kappa = \text{constant}$$

and isotropic, constant mass density

$$\rho^i{}_j = \rho \delta^i_j \qquad \rho = M = \text{constant}$$

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Coordinate transformation

Original coordinates: cylindrical (r, θ, z)
Cloak transformation

$$\bar{r} = \sqrt{r^2 + a^2} \qquad \bar{\theta} = \theta \qquad \bar{z} = z$$

Infinitely extended cloak: $r = 0 \Rightarrow \bar{r} = a$, $r \to \infty \Rightarrow \bar{r} \to r$.

Cylindrical acoustic cloak

Original medium and coordinate transformation

$$C_i^{jkl} = \kappa \delta^j_i g^{kl} \qquad \rho^i{}_j = M\delta^i_j \qquad \bar{r} = \sqrt{r^2 + a^2} \qquad \bar{\theta} = \theta \qquad \bar{z} = z$$

Transformed quantities

$$g_{ij} = \mathrm{diag}\left[1, r^2, 1\right] \qquad \Rightarrow \qquad \bar{g}_{ij} = \mathrm{diag}\left[\frac{\bar{r}^2}{\bar{r}^2 - a^2}, \bar{r}^2 - a^2, 1\right]$$

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Parameters $\boldsymbol{S}'$, ρ', λ' for cloak in original medium, $\boldsymbol{g}' \equiv \boldsymbol{g}$:

$$S'^{-1}_{ij} \rho'^{j}{}_k g^{kl} S'^{-1}_{lm} = M \,\text{diag}\left[\frac{r}{\sqrt{r^2 - a^2}}, \frac{(r^2 - a^2)^{3/2}}{r}, \frac{\sqrt{r^2 - a^2}}{r}\right]$$

$$\lambda' = \frac{\sqrt{r^2 - a^2}}{r} \kappa$$

Cylindrical acoustic cloak

Original medium and coordinate transformation

$$C_i^{jkl} = \kappa\delta_i^j g^{kl} \qquad \rho^i{}_j = M\delta_j^i \qquad \bar{r} = \sqrt{r^2 + a^2} \qquad \bar{\theta} = \theta \qquad \bar{z} = z$$

Intertial cloaking limit

$$g_{ij} = \text{diag}\left[1, r^2, 1\right] \qquad \Rightarrow \qquad \bar{g}_{ij} = \text{diag}\left[\frac{\bar{r}^2}{\bar{r}^2 - a^2}, \bar{r}^2 - a^2, 1\right]$$

Parameters $\boldsymbol{S}'$, ρ', λ' for cloak in original medium, $\boldsymbol{g}' \equiv \boldsymbol{g}$:

$$S'^{ij} = g^{ij} = \text{diag}[1, r^{-2}, 1] \qquad D_i S'^{ij} = 0 \text{ automatically}$$

$$\rho'^i{}_j = M\,\text{diag}\left[\frac{r}{\sqrt{r^2 - a^2}}, \frac{(r^2 - a^2)^{3/2}}{r^3}, \frac{\sqrt{r^2 - a^2}}{r}\right]$$

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Cylindrical acoustic cloak

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Isotropic mass density limit

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$$\rho'^i{}_j = \rho' \delta_j^i$$

$$S'^{-1}_{ij} = \frac{\sqrt{M}}{\sqrt{\rho'}} \mathrm{diag}\left[\frac{\sqrt{r}}{\sqrt[4]{r^2 - a^2}}, \sqrt{r(r^2 - a^2)^{3/2}}, \frac{\sqrt[4]{r^2 - a^2}}{\sqrt{r}}\right]$$

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Cylindrical acoustic cloak

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$$D_i S'^{ij} = 0 \qquad \text{additional condition, DEQ in } \rho'$$

$$\Rightarrow \rho' = cM\sqrt{1 - \frac{a^2}{r^2}} \qquad c \text{ intergration constant}$$

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7 Conclusions

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Wave equation in eigentensor decomposition

Recall the decomposition

$$C_i^{jkl} = g_{im} \sum_{J=1}^{6} \lambda_J S_J^{mj} S_J^{kl}$$

In the momentum balance equation

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In the momentum balance equation with $p_J = -\lambda_J S_J^{mn} e_{mn}$

$$-D_k \left(g_{il} \sum_{J=1}^{6} S_J^{lk} p_J \right) = g_{ij} \rho^j{}_k \ddot{u}^k$$

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Wave equation can be rewritten as coupled DEQ's in scalars of p_J

$$\ddot{p}_K = \sum_J \lambda_K \left(D_i S_K^{ij} - j_K^j \right) \rho_{jk}^{-1} \left(S_J^{kl} D_l + j_J^k \right) p_J \; .$$

with $j_J^i = D_k S_J^{kj}$.

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Transformation of generic wave equation

Start with wave equation

$$\ddot{p}_K = \sum_J \lambda_K \left(D_i S_K^{ij} - j_K^j \right) \rho_{jk}^{-1} \left(S_J^{kl} D_l + j_J^k \right) p_J \ .$$

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Apply transformation $\boldsymbol{x} \rightarrow \bar{\boldsymbol{x}}(x)$ and "remove" covariant derivatives

$$\ddot{p}_K = \sum_J \lambda_K \left(\frac{1}{\sqrt{\bar{g}}} \bar{\partial}_i \sqrt{\bar{g}} \bar{S}^{ij}_K - \bar{j}^j_K \right) \bar{\rho}^{-1}_{jk} \left(\bar{S}^{kl}_J \bar{\partial}_l + \bar{j}^k_J \right) p_J$$

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Reinterpret in coordinates $\boldsymbol{x}'$

$$\ddot{p}_K = \sum_J \frac{\sqrt{g'}}{\sqrt{\bar{g}}} \lambda_K \left(D'_i \bar{S}^{ij}_K - \bar{j}^j_K \right) \frac{\sqrt{\bar{g}}}{\sqrt{g'}} \bar{\rho}^{-1}_{jk} \left(\bar{S}^{kl}_J D'_l + \bar{j}^k_J \right) p_J$$

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TA rules for generic wave equation

Material parameters of TA medium ($\boldsymbol{g}' \equiv \boldsymbol{g}$)

$$C'^{ijkl} = \sum_{K=1}^{6} \lambda'_K S'^{ij}_K S'^{kl}_K \qquad \lambda'_K = \frac{\sqrt{g}}{\sqrt{\bar{g}}}\lambda_K$$

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- Equations are not algebraic

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- Equations are not algebraic
- No guarantee that solutions can be found

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- Equations are not algebraic
- No guarantee that solutions can be found
- Simplifications if (some) $\boldsymbol{S}_J$

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TA rules for generic wave equation

Material parameters of TA medium ($\boldsymbol{g}' \equiv \boldsymbol{g}$)

$$C'^{ijkl} = \sum_{K=1}^{6} \lambda'_K S'^{ij}_K S'^{kl}_K \qquad \lambda'_K = \frac{\sqrt{g}}{\sqrt{\bar{g}}}\lambda_K$$

$$S'^{ik}_K \rho'^{-1}_{kl} S'^{lj}_J = \frac{\sqrt{\bar{g}}}{\sqrt{g}} \bar{S}^{ik}_K \bar{\rho}^{-1}_{kl} \bar{S}^{lj}_J$$

$$D'_i S'^{ij}_K = 0$$

- Equations are not algebraic
- No guarantee that solutions can be found
- Simplifications if (some) $\boldsymbol{S}_J$
  - obey $D_i S^{ij}_J = 0 \Rightarrow j^i_J = 0$

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TA rules for generic wave equation

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$$S'^{-1}_{K\,kj} D'_i S'^{ij}_K = \bar{S}^{-1}_{K\,kj} \bar{D}_i \bar{S}^{ij}_K$$

- Equations are not algebraic
- No guarantee that solutions can be found
- Simplifications if (some) $\boldsymbol{S}_J$
  - obey $D_i S^{ij}_J = 0 \Rightarrow j^i_J = 0$
  - are invertible

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Outline

1. Geometric ('Ray') Transformation Acoustics
   - Possible approaches to transformation acoustics
   - Parallel transport and geodesics
   - Example: acoustic ray cloak
2. Linear elastodynamics
3. Equations of motion for linearised elastodynamics
4. Transformation acoustics for special materials
   - Intertial transformation acoustics
   - Pentamode transformation acoustics
5. Example: cylindrical acoustic cloak
6. TA beyond pentamodes
7. Conclusions

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Conclusions

Ray TA

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- Approximation, not wave exact

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Generic TA

- Generic wave equation becomes coupled DEQs of scalars
- Material equations in general extremely complicated
- No solution guaranteed

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