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Electromagnetic Media with no Dispersion Equation

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Contents

The present paper considers the possibility of defining electromagnetic media in which a plane wave is not restricted by a dispersion equation (Fresnel equation)

- Introduction: plane waves and dispersion equations
- Example of a medium with no dispersion equation
- Four-Dimensional Formalism applied in analysis
- Classes of media with no Dispersion Equation defined
- Discussion and Conclusion

Introduction

Plane Wave in Linear Medium

- Time-harmonic plane wave in a linear, homogeneous, time invariant medium is defined by fields of the form

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}\exp(-j\mathbf{k}\cdot\boldsymbol{r}), \qquad \mathbf{H}(\mathbf{r}) = \mathbf{H}\exp(-j\mathbf{k}\cdot\boldsymbol{r})$$

- Eliminating fields, Maxwell equations yield

$$\overline{\overline{\mathsf{D}}}(\mathbf{k})\cdot\mathbf{E} = 0$$

- For $\mathbf{E} \neq 0$ wave vector $\mathbf{k}$ is restricted by dispersion equation

$$D(\mathbf{k}) = \det\overline{\overline{\mathsf{D}}}(\mathbf{k}) = 0$$

- Algebraic equation of the 4th order in general, coefficients depend on the medium parameters

Classifying Dispersion Equations

- Dispersion equation defines a surface in the wave-vector $\mathbf{k}$ space as a function of the unit vector $\mathbf{u}$

$$D(\mathbf{k}) = 0 \quad \Rightarrow \quad \mathbf{k} = \mathbf{u}k(\mathbf{u})$$

- Nature of the surface $k = k(\mathbf{u})$ depends on the medium
  1. General medium: quartic surface
  2. Special case: two quadratic surfaces ("decomposable medium")
  3. More special case: single quadratic surface ("nonbirefringent medium")
  4. $D(\mathbf{k}) = 0$ satisfied identically for any $\mathbf{k}$, no dispersion equation. Choice of wave vector $\mathbf{k}$ is not restricted.
- Item 4 is associated to "media with no dispersion equation"

Medium with No Dispersion Equation

- As an example, consider a medium defined by

$$\mathbf{D} = (\overline{\overline{\alpha}} + M\overline{\overline{\mathsf{I}}}) \cdot \mathbf{B} + \mathbf{c} \times \mathbf{E}$$

$$\mathbf{H} = \mathbf{g} \times \mathbf{B} + (\overline{\overline{\alpha}}^T - M\overline{\overline{\mathsf{I}}}) \cdot \mathbf{E}$$

where $\overline{\overline{\alpha}}$ is a dyadic, $\mathbf{c}$ and $\mathbf{g}$ are two vectors and M is a scalar,

- Equation for field $\mathbf{E}$ becomes

$$\overline{\overline{\mathsf{D}}}(\mathbf{k}) \cdot \mathbf{E} = \mathbf{q}(\mathbf{k}) \times \mathbf{E} = 0$$

$$\mathbf{q}(\mathbf{k}) = (\mathbf{g} \cdot \mathbf{k} - \omega \mathrm{tr}\overline{\overline{\alpha}})\mathbf{k} + \omega \mathbf{k} \cdot \overline{\overline{\alpha}} + \omega^2 \mathbf{c}$$

- Dispersion equation is satisfied identically:

$$D(\mathbf{k}) = \det\overline{\overline{\mathsf{D}}}(\mathbf{k}) = \det(\mathbf{q}(\mathbf{k}) \times \overline{\overline{\mathsf{I}}}) = 0 \quad \text{for all } \mathbf{k}$$

- The medium does not have a dispersion equation!

Boundary Conditions from Interface Conditions

- Special case $\overline{\overline{\alpha}} = 0, \mathbf{c} = 0, \mathbf{g} = 0$ yields PEMC medium

$$\mathbf{D} = M\mathbf{B}, \quad \mathbf{H} = -M\mathbf{E}$$

- Also PMC ($M = 0$) and PEC ($|M| \to \infty$) media do not have a dispersion equation
- Media with no dispersion equation may define useful boundary conditions!
- PEMC boundary: $\mathbf{n} \times (\mathbf{H} + M\mathbf{E}) = 0, \ \mathbf{n} \cdot (\mathbf{D} - M\mathbf{B}) = 0$
- Uniaxial medium yields DB boundary conditions $\mathbf{n} \cdot \mathbf{B} = 0$, $\mathbf{n} \cdot \mathbf{D} = 0$
- Other: SH (Soft-and-Hard) and SHDB boundary conditions

Four-Dimensional Formalism

EM Field Equations

- Maxwell equations outside sources

$$\mathbf{d} \wedge \boldsymbol{\Phi} = 0, \quad \mathbf{d} \wedge \boldsymbol{\Psi} = 0$$

- Field two-forms in spatial and temporal components ($\varepsilon_4 = \mathbf{d}ct$)

$$\boldsymbol{\Phi} = \mathbf{B} + \mathbf{E} \wedge \varepsilon_4, \quad \boldsymbol{\Psi} = \mathbf{D} - \mathbf{H} \wedge \varepsilon_4$$

- Plane-wave fields for $\mathbf{x} = \mathbf{r} + \mathbf{e}_4 ct, \quad \boldsymbol{\nu} = \boldsymbol{\beta} + \varepsilon_4 \omega / c$

$$\boldsymbol{\Phi}(\mathbf{x}) = \boldsymbol{\Phi} \exp(\boldsymbol{\nu}|\mathbf{x}), \quad \boldsymbol{\Psi}(\mathbf{x}) = \boldsymbol{\Psi} \exp(\boldsymbol{\nu}|\mathbf{x})$$

- Representation in terms of potential one-form ϕ

$$\boldsymbol{\nu} \wedge \boldsymbol{\Phi} = 0 \quad \Rightarrow \quad \boldsymbol{\Phi} = \boldsymbol{\nu} \wedge \phi$$

I.V. Lindell *Differential Forms in Electromagnetics*, IEEE Press 2004.

Medium Equations

- Medium bidyadic $\overline{\overline{\mathsf{M}}}$ maps two-forms to two-forms

$$\boldsymbol{\Psi} = \overline{\overline{\mathsf{M}}}|\boldsymbol{\Phi}$$

- Corresponds to four spatial dyadics

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\alpha}} & \overline{\overline{\epsilon}}' \\ \overline{\overline{\mu}}^{-1} & \overline{\overline{\beta}} \end{pmatrix} | \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}$$

- Modified medium bidyadic $\overline{\overline{\mathsf{M}}}_m$ maps two-forms to bivectors

$$\mathbf{e}_N\lfloor\boldsymbol{\Psi} = \overline{\overline{\mathsf{M}}}_m|\boldsymbol{\Phi}, \quad \overline{\overline{\mathsf{M}}}_m = \mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}$$

- Quadrivector $\mathbf{e}_N = \mathbf{e}_{1234} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$

Plane-Wave Equations in 4D

- Maxwell equation $\Rightarrow$ equation for potential one-form ϕ

$$\boldsymbol{\nu} \wedge \boldsymbol{\Psi} = \boldsymbol{\nu} \wedge \overline{\overline{\mathsf{M}}}|\boldsymbol{\Phi} = \boldsymbol{\nu} \wedge \overline{\overline{\mathsf{M}}}|(\boldsymbol{\nu} \wedge \boldsymbol{\phi}) = (\boldsymbol{\nu} \wedge \overline{\overline{\mathsf{M}}}\lfloor \boldsymbol{\nu})|\boldsymbol{\phi} = 0$$

- Dispersion dyadic $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})$ maps one-forms to vectors

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})|\boldsymbol{\phi} = 0, \quad \overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = \overline{\overline{\mathsf{M}}}_m \lfloor\lfloor \boldsymbol{\nu}\boldsymbol{\nu} = -\boldsymbol{\nu}\rfloor\overline{\overline{\mathsf{M}}}_m\lfloor \boldsymbol{\nu}$$

- Because also $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})|\boldsymbol{\nu} = 0$, rank of $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})$ must be < 3:

$$\overline{\overline{\mathsf{D}}}^{(3)}(\boldsymbol{\nu}) = \frac{1}{6}\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})^{\wedge}_{\wedge}\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})^{\wedge}_{\wedge}\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = 0$$

- Equivalent scalar dispersion equation of 4th order in $\boldsymbol{\nu}$

$$D(\boldsymbol{\nu}) = \frac{1}{6}\varepsilon_N\varepsilon_N||(\overline{\overline{\mathsf{M}}}_m{}^{\wedge}_{\wedge}(\boldsymbol{\nu}\boldsymbol{\nu}\rfloor\rfloor(\overline{\overline{\mathsf{M}}}_m{}^{\wedge}_{\wedge}(\boldsymbol{\nu}\boldsymbol{\nu}\rfloor\rfloor\overline{\overline{\mathsf{M}}}_m)))) = 0$$

Media With no Dispersion Equation

Dispersion Dyadic

- Dispersion dyadic satisfying $\overline{\overline{\mathsf{D}}}^{(3)}(\boldsymbol{\nu}) = 0$ can be expanded as

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = \overline{\overline{\mathsf{M}}}_m \lfloor\lfloor \boldsymbol{\nu}\boldsymbol{\nu} = \mathbf{ac} + \mathbf{bd}$$

- Assume $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})$ of rank 2 for all $\boldsymbol{\nu} \quad \Rightarrow \quad (\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d}) \neq 0$
- Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are functions of the wave one-form $\boldsymbol{\nu}$
- $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})$ is quadratic function of $\boldsymbol{\nu} \Rightarrow$ Four basic possibilities.
  1. $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ linear functions of $\boldsymbol{\nu}$
  2. $\mathbf{a}, \mathbf{b}$ quadratic functions, $\mathbf{c}, \mathbf{d}$ independent of $\boldsymbol{\nu}$
  3. $\mathbf{a}, \mathbf{d}$ quadratic, $\mathbf{b}, \mathbf{c}$ independent of $\boldsymbol{\nu}$
  4. $\mathbf{a}$ quadratic, $\mathbf{b}, \mathbf{d}$ linear functions, $\mathbf{c}$ independent of $\boldsymbol{\nu}$
- Other possibilities can be reduced to these four cases

Case 1

- Assume vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are linear functions of $\boldsymbol{\nu}$

$$\boldsymbol{\nu}|\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = -(\boldsymbol{\nu}\wedge\boldsymbol{\nu})|\overline{\overline{\mathsf{M}}}_m\rfloor\boldsymbol{\nu} = 0 \quad \text{for all } \boldsymbol{\nu}$$

$$\Rightarrow \quad (\boldsymbol{\nu}|\mathbf{a})\mathbf{c} + (\boldsymbol{\nu}|\mathbf{b})\mathbf{d} = 0 \quad \text{for all } \boldsymbol{\nu}$$

- Because $\mathbf{c}, \mathbf{d}$ are linearly independent $(\mathbf{c}\wedge\mathbf{d} \neq 0)$

$$\Rightarrow \quad \boldsymbol{\nu}|\mathbf{a} = \boldsymbol{\nu}|\mathbf{b} = 0 \quad \text{for all } \boldsymbol{\nu}$$

- Vectors $\mathbf{a}, \mathbf{b}$ can be expressed in terms of some bivectors $\mathbf{A}, \mathbf{B}$ as

$$\mathbf{a} = \mathbf{A}\lfloor\boldsymbol{\nu}, \quad \mathbf{b} = \mathbf{B}\lfloor\boldsymbol{\nu}$$

Case 1 cont'd

- Similarly, vectors $\mathbf{c}, \mathbf{d}$ can be expressed in terms of some bivectors $\mathbf{C}, \mathbf{D}$ as

$$\mathbf{c} = \mathbf{C}\lfloor\boldsymbol{\nu}, \quad \mathbf{d} = \mathbf{D}\lfloor\boldsymbol{\nu}$$

- Dispersion dyadic of Case 1 has the representation

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = \overline{\overline{\mathsf{M}}}_m\lfloor\lfloor\boldsymbol{\nu}\boldsymbol{\nu} = (\mathbf{AC} + \mathbf{BD})\lfloor\lfloor\boldsymbol{\nu}\boldsymbol{\nu} \quad \text{for all } \boldsymbol{\nu}$$

- Apply property: if a bidyadic $\overline{\overline{\mathsf{A}}}$ satisfies $\overline{\overline{\mathsf{A}}}\lfloor\lfloor\boldsymbol{\nu}\boldsymbol{\nu} = 0$ for all $\boldsymbol{\nu}$, it must be a multiple of the ("unit") bidyadic $\mathbf{e}_N\lfloor\bar{\bar{\mathsf{I}}}^{(2)T}$.
- For Case 1 modified medium bidyadic must be of the form

$$\overline{\overline{\mathsf{M}}}_m = \mathbf{AC} + \mathbf{BD} + M\mathbf{e}_N\lfloor\bar{\bar{\mathsf{I}}}^{(2)T}$$

- $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are arbitrary bivectors and M any scalar.

Case 2

- Assume $\mathbf{a}, \mathbf{b}$ are quadratic functions of $\boldsymbol{\nu}$ while $\mathbf{c}, \mathbf{d}$ are independent of $\boldsymbol{\nu}$
- After some algebraic reasoning one can show that there are two main Case 2 solutions:
  1. "Skewon-axion medium" $\overline{\overline{\mathsf{M}}}_m = \overline{\overline{\mathsf{A}}} + M\mathbf{e}_N\lfloor\bar{\bar{\mathsf{I}}}^{(2)T}$
     where $\overline{\overline{\mathsf{A}}}$ is any antisymmetric bidyadic
     Number of parameters $15(\overline{\overline{\mathsf{A}}}) + 1(M) = 16$
  2. "P-axion medium" $\overline{\overline{\mathsf{M}}}_m = \overline{\overline{\mathsf{P}}}^{(2)T} + M\mathbf{e}_N\lfloor\bar{\bar{\mathsf{I}}}^{(2)T}$
     where $\overline{\overline{\mathsf{P}}}$ is any dyadic mapping vectors to vectors
     Number of parameters $16(\overline{\overline{\mathsf{P}}}) + 1(M) = 17$

Case 2 in Gibbsian form

- Medium equations in terms of Gibbsian 3D vectors and dyadics
  1. "Skewon-axion medium"

$$\mathbf{D} = (\overline{\overline{\alpha}} + M\overline{\overline{\mathsf{I}}}) \cdot \mathbf{B} + \mathbf{c} \times \mathbf{E}$$

$$\mathbf{H} = \mathbf{g} \times \mathbf{B} + (\overline{\overline{\alpha}}^T - M\overline{\overline{\mathsf{I}}}) \cdot \mathbf{E}$$

Number of parameters $9(\overline{\overline{\alpha}}) + 3(\mathbf{c}) + 3(\mathbf{g}) + 1(M) = 16$

  2. "P-axion medium"

$$\mathbf{D} = (\overline{\overline{\beta}}^{(2)} + M\overline{\overline{\mathsf{I}}}) \cdot \mathbf{B} + (\mathbf{q} \times \overline{\overline{\beta}}) \cdot \mathbf{E}$$

$$\mathbf{H} = (\overline{\overline{\beta}} \times \mathbf{p}) \cdot \mathbf{B} + (\mathbf{qp} + p\overline{\overline{\beta}} - M\overline{\overline{\mathsf{I}}}) \cdot \mathbf{E}$$

Number of parameters $9(\overline{\overline{\beta}}) + 3(\mathbf{q}) + 3(\mathbf{p}) + 1(p) + 1(M) = 17$

- Skewon-axion medium equals the example in the introduction

Discussion and Conclusion

Other Solutions?

- One can show that Case 3 and Case 4 do not yield new solutions
- The medium equation in the inverse form $\mathbf{\Phi} = \overline{\overline{\mathsf{N}}}|\mathbf{\Psi}$ yields the same dispersion equation. No new media without dispersion equation will emerge because
  - the inverse of a Case 1 bidyadic $\overline{\overline{\mathsf{M}}}$ is a Case 1 bidyadic $\overline{\overline{\mathsf{N}}}$
  - the inverse of a general P-axion bidyadic $\overline{\overline{\mathsf{M}}}$ is a general P-axion bidyadic $\overline{\overline{\mathsf{N}}}$
  - the inverse of a skewon-axion bidyadic $\overline{\overline{\mathsf{M}}}$ is a special P-axion bidyadic $\overline{\overline{\mathsf{N}}}$ and conversely
- Dispersion dyadic $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})$ of rank 1 yields special cases of the previous solutions of rank 2
- However, a decisive proof for Case 1 and Case 2 solutions being the only ones has not (yet) been found

Conclusion

- Since various studies have shown that there exist media with no dispersion equation, a more systematic study to define them was made
- 4D formalism was applied for conciseness of notation
- Three classes of media with no dispersion equation was found through the analysis
- Case 2 media were known from previous analyses, Case 1 medium class appears to be new
- The solutions may have application as defining novel boundary conditions at the interface
- More information: I.V. Lindell, A. Favaro "Electromagnetic media with no dispersion equation, *Progress in Electromagnetics Research PIER B* vol.51, pp.269–289, 2013.

Appendix: Hehl-Obukhov Decomposition

- Consider medium equation in Gibbsian 3D dyadics

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\overline{\overline{\epsilon}}' & \overline{\overline{\alpha}} \\ -\overline{\overline{\beta}} & \overline{\overline{\mu}}^{-1} \end{pmatrix} \cdot \begin{pmatrix} -\mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

- Hehl-Obukhov decomposition of medium dyadics in three parts

$$\begin{pmatrix} -\overline{\overline{\epsilon}}' & \overline{\overline{\alpha}} \\ -\overline{\overline{\beta}} & \overline{\overline{\mu}}^{-1} \end{pmatrix} = \begin{pmatrix} -\overline{\overline{\epsilon}}'_1 & \overline{\overline{\alpha}}_1 \\ -\overline{\overline{\beta}}_1 & \overline{\overline{\mu}}_1^{-1} \end{pmatrix} + \begin{pmatrix} -\overline{\overline{\epsilon}}'_2 & \overline{\overline{\alpha}}_2 \\ -\overline{\overline{\beta}}_2 & \overline{\overline{\mu}}_2^{-1} \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & \overline{\overline{\mathsf{I}}} \\ \overline{\overline{\mathsf{I}}} & 0 \end{pmatrix}$$

1. Principal part: symmetric dyadic matrix, $\mathrm{tr}\overline{\overline{\alpha}}_1 = 0$
2. Skewon part: antisymmetric dyadic matrix,
3. Axion part: $\overline{\overline{\alpha}}_3 = -\overline{\overline{\beta}}_3 = \alpha_3 \overline{\overline{\mathsf{I}}}$, $\overline{\overline{\epsilon}}'_3 = 0, \overline{\overline{\mu}}_3^{-1} = 0$

F.W. Hehl, Yu.N. Obukhov, *Foundations of Classical Electrodynamics*, Boston: Birkhäuser, 2003.