Pre-metric electrodynamics, electric-magnetic duality & closure relations.

A. Favaro, L. Bergamin, I.V. Lindell, Y.N. Obukhov.

Respectively at:

- Department of Physics, Imperial College London, UK.
- KB&P GmbH, Bern, Switzerland.
- Department of Radio Science and Engineering, Aalto University, Finland.
- Institute for Theoretical Physics, University of Cologne, Germany.

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Outline

Basics of pre-metric electrodynamics

- Representation of fields as differential forms or tensors.
- Energy-momentum tensor (3-form) and field invariants.
- Assume the medium is **local** (dispersionless) & **linear**.
- Explain principal+skewon+axion split of the medium.

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Solve two closure relations explicitly (invertible medium)

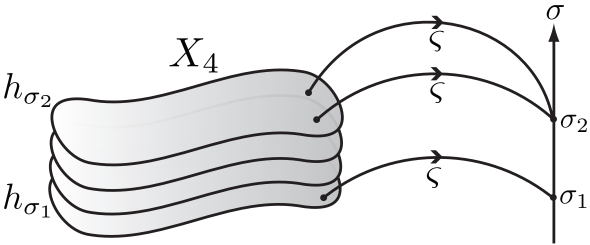
- Invertible media: solve 2 (out of 4) closures. Re-derive.

Electromagnetic fields as differential forms

Fundamental fields of Electromagnetism as differential forms

$$J = -\mathrm{d}\sigma \wedge j + \rho \qquad \text{twisted 3-form,}$$
$$H = \ \mathrm{d}\sigma \wedge \mathcal{H} + \mathcal{D}\,, \qquad \text{twisted 2-form,}$$
$$F = -\mathrm{d}\sigma \wedge E + B\,, \qquad \text{ordinary 2-form.}$$

Fields $\{j, \rho, \mathcal{H}, \mathcal{D}, E, B\}$ obtained by slicing spacetime X_4, as



Embedded submanifolds h_σ are space, σ is topological time.
No metric or connection needed: pre-metric electrodynamics.

Electromagnetic fields as antisymmetric tensors

Current density $J_{\alpha\beta\gamma}$, field excitation $H_{\alpha\beta}$, field strength $F_{\alpha\beta}$:

$$J_{\alpha\beta\gamma} = J_{[\alpha\beta\gamma]}\,, \qquad H_{\alpha\beta} = H_{[\alpha\beta]}\,, \qquad F_{\alpha\beta} = F_{[\alpha\beta]}\,.$$

In tensor formalism, the familiar fields $\{j, \rho, \mathcal{H}, \mathcal{D}, E, B\}$ read

$$\begin{aligned} J_{0ab} &= -j_{ab}\,, & J_{abc} &= \rho_{abc}\,, \\ H_{0a} &= \quad \mathcal{H}_a\,, & H_{ab} &= \mathcal{D}_{ab}\,, \\ F_{0a} &= -E_a\,, & F_{ab} &= B_{ab}\,, \end{aligned}$$

with indices $\{\alpha, \beta, \cdots = 0, 1, 2, 3\}$ and $\{a, b, \cdots = 1, 2, 3\}$.

Maxwell's equations: use differential forms or tensors

Note: Maxwell's equations require no metric or connection

$$\begin{aligned} \mathrm{d}H &= J\,, & \mathrm{d}F &= 0\,, \\ \partial_{[\alpha}H_{\beta\gamma]} &= J_{\alpha\beta\gamma}\,, & \partial_{[\alpha}F_{\beta\gamma]} &= 0\,. \end{aligned}$$

Pair of antisymmetric indices → Collective label

Example: indices of $F_{\alpha\beta}$ and $H_{\alpha\beta}$ are antisymmetric. Hence, $F_{\alpha\beta}$ and $H_{\alpha\beta}$ have 6 independent entries. Label them as

$$\{[\alpha\beta] = [01], [02], [03], [23], [31], [12]\} \to \{I = 1, 2, \ldots, 6\}.$$

Thereby, represent $H_{\alpha\beta}$ and $F_{\alpha\beta}$ as columns with 6 entries

$$H_I = \begin{bmatrix} H_{01} \\ H_{02} \\ H_{03} \\ H_{23} \\ H_{31} \\ H_{12} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{D}_{23} \\ \mathcal{D}_{31} \\ \mathcal{D}_{12} \end{bmatrix}, \qquad F_I = \begin{bmatrix} F_{01} \\ F_{02} \\ F_{03} \\ F_{23} \\ F_{31} \\ F_{12} \end{bmatrix} = \begin{bmatrix} -E_1 \\ -E_2 \\ -E_3 \\ B_{23} \\ B_{31} \\ B_{12} \end{bmatrix}.$$

nice separation of electric and magnetic. **Summary**: pair of antisymmetric indices → collective label $\{I, J, \ldots = 1, \ldots, 6\}$.

Minkowski (pre-metric) energy-momentum tensor

Using **tensors**, Minkowski (pre-metric) energy-momentum:

$$\mathcal{T}_\alpha{}^\beta = \frac{1}{4}\epsilon^{\beta\mu\rho\sigma}(H_{\alpha\mu}F_{\rho\sigma} - F_{\alpha\mu}H_{\rho\sigma}).$$

Using **differential forms**, energy-momentum transfer is encoded by means of a twisted covector-valued 3-form

$$\Sigma_\alpha = \frac{1}{2}\left[F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)\right],$$

where $\{e_\alpha\}$ is the frame. Space+time decomposition leads to

$$\left[\begin{array}{c|c} \mathcal{T}_0{}^0 & \mathcal{T}_0{}^b \\ \hline \mathcal{T}_a{}^0 & \mathcal{T}_a{}^b \end{array}\right] = \left[\begin{array}{c|c} u & s^b \\ \hline -p_a & -S_a{}^b \end{array}\right]$$

where u is the energy density, s^b is the energy flux density, p_a is momentum density and $S_a{}^b$ is momentum flux density. Similar decomposition found when using differential forms.

Invariants of the electromagnetic field

A **4-form** in spacetime has 1 independent component, it encodes an invariant. Use H and F to build the invariants

$$
\begin{aligned}
I_1 &= F \wedge H = \quad \mathrm{d}\sigma \wedge (B \wedge \mathcal{H} - E \wedge \mathcal{D}), \\
I_2 &= F \wedge F = -2\mathrm{d}\sigma \wedge (B \wedge E), \\
I_3 &= H \wedge H = \quad 2\mathrm{d}\sigma \wedge (\mathcal{H} \wedge \mathcal{D}).
\end{aligned}
$$

There exists a fourth invariant $I_4 = A \wedge J$, but leave aside. Setting one of $\{I_1, I_2, I_3\}$ to be zero, is a statement about the **configuration** of the fields that holds true in any frame.

4-dim.	3-dim. (pre-metric)	3-dim. (post-metric)
$I_1 = 0$	$\frac{1}{2} B \wedge \mathcal{H} = \frac{1}{2} E \wedge \mathcal{D}$	$\frac{1}{2}\vec{B} \cdot \vec{H} = \frac{1}{2}\vec{E} \cdot \vec{D}$
$I_2 = 0$	$B \wedge E = 0$	$\vec{B} \cdot \vec{E} = 0$
$I_3 = 0$	$\mathcal{H} \wedge \mathcal{D} = 0$	$\vec{D} \cdot \vec{H} = 0$

For **plane** waves, $I_1 = I_2 = I_3 = 0$; but not true in **general**.

Local and linear media

Local and linear media

- Given a point p in spacetime, the medium response is local if $H\big|_p$ is a function of $F\big|_p$ only. In other words:

$$H = \kappa(F)\,, \qquad \text{(local constitutive law)},$$

where κ is a map from ordinary to twisted 2-forms.

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- In particular, the medium response is linear whenever:

$$\kappa(a\Psi_1 + b\Psi_2) = a\kappa(\Psi_1) + b\kappa(\Psi_2)\,, \qquad \text{(linear law)},$$

for any 2-forms $\{\Psi_1, \Psi_2\}$ and functions $\{a, b\}$. Then,

$$H_{\alpha\beta} = \frac{1}{2}\kappa_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}\,, \qquad \text{(tensor indices)},$$

$$H_I = \kappa_I{}^J F_J\,, \qquad \text{(6-dim indices)}.$$

Clearly, Einstein's summation convention is employed.

Local an linear media (space+time split)

In terms of $\{\mathcal{H}, \mathcal{D}, E, B\}$ the local and linear law is given by:

$$\mathcal{H}_a = \beta_a{}^c E_c + \frac{1}{2}(\mu^{-1})_a{}^{cd} B_{cd}\,,$$
$$\mathcal{D}_{ab} = \varepsilon'_{ab}{}^c E_c + \frac{1}{2}\alpha_{ab}{}^{cd} B_{cd}\,.$$

as seen in Lindell's book (IEEE, 2004). More specifically:

$$\beta_a{}^c := -\kappa_{0a}{}^{0c}\,, \qquad (\mu^{-1})_a{}^{cd} := \kappa_{0a}{}^{cd}\,,$$
$$\varepsilon'_{ab}{}^c := -\kappa_{ab}{}^{0c}\,, \qquad \alpha_{ab}{}^{cd} := \kappa_{ab}{}^{cd}\,.$$

When $\kappa_I{}^J$ represented as 6×6 matrix, one attains that

$$\left[\begin{array}{ccc|ccc}
-\beta_1{}^1 & -\beta_1{}^2 & -\beta_1{}^3 & (\mu^{-1})_1{}^{23} & (\mu^{-1})_1{}^{31} & (\mu^{-1})_1{}^{12} \\
-\beta_2{}^1 & -\beta_2{}^2 & -\beta_2{}^3 & (\mu^{-1})_2{}^{23} & (\mu^{-1})_2{}^{31} & (\mu^{-1})_2{}^{12} \\
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\hline
-\varepsilon'_{23}{}^1 & -\varepsilon'_{23}{}^2 & -\varepsilon'_{23}{}^3 & \alpha_{23}{}^{23} & \alpha_{23}{}^{31} & \alpha_{23}{}^{12} \\
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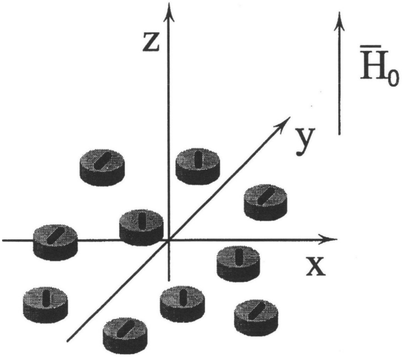
\end{array}\right].$$

Example of magneto-electric metamaterial



- **Figure**: Tretyakov et al., J. Electromagnet Wave, 1998.
- **Idea**: Kamenetskii, Microw. Opt. Techn. Lett., 1996.
- **Medium**: Ellipsoidal ferrite inclusions subject to fixed magnetic $\bar{H}_0$. Each inclusion is fitted with metal strip.
- Magnetic field **input** $\Rightarrow$ inclusions' magnetic resonance $\Rightarrow$ currents in metal strips $\Rightarrow$ an electric field **output**.

The "bar conjugate" of the medium response

In preparation for decomposing κ, define the "bar conjugate"

$$\bar{\kappa}_{\alpha\beta}{}^{\mu\nu} = \frac{1}{4}\hat{\epsilon}_{\alpha\beta\rho\sigma}(\kappa_{\eta\theta}{}^{\rho\sigma})\epsilon^{\eta\theta\mu\nu} \,.$$

Note: $\kappa_{\alpha\beta}{}^{\mu\nu}$ and $\bar{\kappa}_{\alpha\beta}{}^{\mu\nu}$ have **same** domain and co-domain.

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- The transposed map $\kappa^{\mathbf{t}}$: bivectors $\to$ twisted bivectors,

$$B := \kappa^{\mathbf{t}}(A) \quad \text{stands for} \quad B^{\alpha\beta} := \frac{1}{2}\kappa_{\mu\nu}{}^{\alpha\beta}A^{\mu\nu}.$$

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For $\diamondsuit_2$ and $\hat{\diamondsuit}_2$ see Greub (1967), Kurz & Heumann (2010).

The "bar conjugate" (continued)

Bar conjugate is the composition of maps $\bar{\kappa} := \hat{\Diamond}_2 \circ \kappa^{\mathbf{t}} \circ \Diamond_2$,

$$\text{2-form} \xrightarrow{\Diamond_2} \text{bivector d.} \xrightarrow{\kappa^{\mathbf{t}}} \text{tw. bivector d.} \xrightarrow{\hat{\Diamond}_2} \text{tw. 2-form}$$

where "tw." means twisted and "d." means density. Crucial to note that κ and $\bar{\kappa}$ have the same domain and co-domain. Caveat: $\Diamond_2$ and $\hat{\Diamond}_2$ yield opposite density weights, +1 & -1.

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- If κ' is the composition of two operators ($\kappa' = \kappa_1 \circ \kappa_2$),

$$\bar{\kappa}' = \bar{\kappa}_2 \circ \bar{\kappa}_1 \,.$$

Principal+Skewon+Axion decomposition

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Symmetric & Antisymmetric contributions

Split κ in a **symmetric** and an **antisymmetric** part with respect to the bar conjugate, $\kappa = {}^{(+)}\kappa + {}^{(-)}\kappa$. In particular,

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\begin{aligned}
{}^{(+)}\bar{\kappa} &= +\,{}^{(+)}\kappa\,, \\
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a) Split the **symmetric** piece ${}^{(+)}\kappa$ in a **traceless** part and a **trace** contribution. Thereby, obtain ${}^{(+)}\kappa = {}^{(1)}\kappa + {}^{(3)}\kappa$.

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c) **Principal-Skewon-Axion** split $\kappa = {}^{(1)}\kappa + {}^{(2)}\kappa + {}^{(3)}\kappa$,

$$ {}^{(1)}\bar{\kappa} = + {}^{(1)}\kappa \,, \qquad \mathrm{tr}[{}^{(1)}\kappa] = 0 \,, $$
$$ {}^{(2)}\bar{\kappa} = - {}^{(2)}\kappa \,, \qquad \mathrm{tr}[{}^{(2)}\kappa] \equiv 0 \,, $$
$$ {}^{(3)}\bar{\kappa} = + {}^{(3)}\kappa \,, \qquad \mathrm{tr}[{}^{(3)}\kappa] = \mathrm{tr}(\kappa) \,. $$

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Preview: Four closure relations

In solving some electromagnetic problems (examples later), one encounters the so-called **closure relations**, restricting κ.

Closure relations: Pure and Mixed

- The **pure** closure relations are:

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Crucially, the true scalars in red are allowed to **vanish** (at least for the moment), and to take **any sign**. We consider few physical questions in which **closure relations appear**.

Electric-magnetic reciprocity

- Given a twisted scalar $\zeta \neq 0$, with dimensions of inverse resistance, define the electric-magnetic reciprocity as:

$$\left.\begin{matrix}\mathcal{H}' \\ \mathcal{D}'\end{matrix}\right\} = H' = +\zeta F = \left\{\begin{matrix} -\zeta E \\ +\zeta B\end{matrix}\right.$$

$$\left.\begin{matrix}E' \\ B'\end{matrix}\right\} = F' = -\frac{1}{\zeta} H = \left\{\begin{matrix} +\frac{1}{\zeta}\mathcal{H} \\ -\frac{1}{\zeta}\mathcal{D}\end{matrix}\right.$$

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$$H' = \kappa(F') \;\; \Rightarrow \;\; \zeta F = \kappa(-\zeta^{-1} H) \;\; \Rightarrow \;\; F = -\zeta^{-2}\kappa(H).$$

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Electric-magnetic reciprocity → a closure relation

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Electric-magnetic reciprocity $\rightarrow$ a closure relation

- Electric-magnetic reciprocal media obey $\kappa \circ \kappa = -\zeta^2 \mathrm{Id}$. That is, they are solutions of the **pure** closure relation

$$\kappa \circ \kappa = \frac{1}{6}\mathrm{tr}(\kappa \circ \kappa)\mathrm{Id}\,,$$

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- See: Peres (1962), Toupin (1965), Schönberg (1971), Obukhov and Hehl (1999), Rubilar (2002), Dahl (2011).
- Consider another physical question leading to above closure relation, but with $\mathrm{tr}(\kappa \circ \kappa)$ entirely **arbitrary**.

The special linear SL(2,$\mathbb{R}$) reciprocity

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The special linear SL(2,$\mathbb{R}$) reciprocity

Start from arbitrary linear reciprocity

Consider an arbitrary matrix mapping $(H; F)$ into $(H'; F')$ as

$$\begin{bmatrix} H' \\ F' \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} \begin{bmatrix} H \\ F \end{bmatrix}$$

- $\{C_{00}, C_{11}\}$ twist-free and dimensionless.
- $\{C_{01}, C_{10}\}$ twisted, with $[C_{01}] = [C_{10}]^{-1} = [\text{resistance}]^{-1}$.

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- **Special linear** reciprocity has a physical importance.

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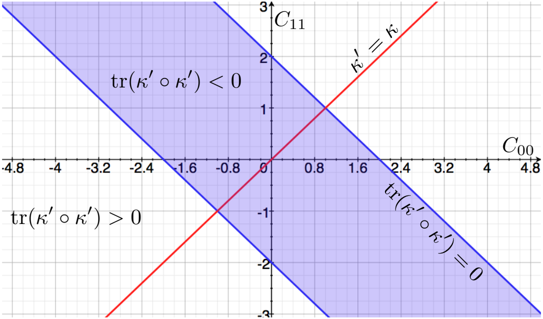


- SL(2, $\mathbb{R}$) reciprocal media obey the **pure** closure relation

$$\kappa' \circ \kappa' = \frac{1}{6}\mathrm{tr}(\kappa' \circ \kappa')\mathrm{Id},$$

provided one introduces a "modified" map κ' such that

$$\kappa' := \kappa + \left(\frac{C_{11} - C_{00}}{2C_{10}}\right)\mathrm{Id} \quad \text{and} \quad \mathrm{tr}(\kappa' \circ \kappa') = \frac{(C_{11} + C_{00})^2 - 4}{4C_{10}^2}.$$

- The factor $\mathrm{tr}(\kappa' \circ \kappa')$ can take **any sign**, or even **vanish**.

Mixed closure relation when invariants $I_3 = \eta I_2$

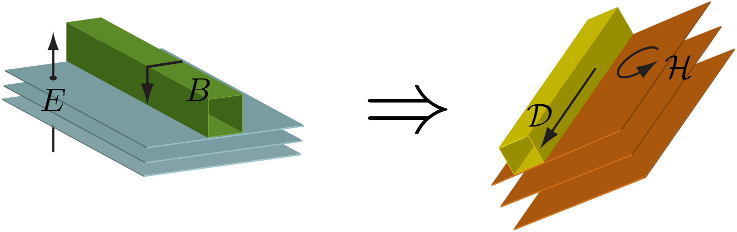
- Look for medium such that, for **every** choice of $\{H, F\}$,

$$I_3 = \eta I_2 \,, \qquad \text{that is} \qquad H \wedge H = \eta F \wedge F \,,$$

In terms of 3-dimensional fields, for **every** $\{\mathcal{H}, \mathcal{D}, E, B\}$,

$$\mathcal{H} \wedge \mathcal{D} = -\eta B \wedge E \,, \qquad \text{(pre-metric)},$$
$$\vec{H} \cdot \vec{D} = -\eta \vec{B} \cdot \vec{E} \,, \qquad \text{(post-metric)}.$$

- Consequence: if $B \wedge E = 0$, one has $\mathcal{H} \wedge \mathcal{D} \equiv 0$ trivially.



From $I_3 = \eta I_2$ ($\forall$ fields) to mixed closure relation.

- Demand $H \wedge H = \eta F \wedge F$ for **every** choice of H and F. Local & linear media: $\kappa(F) \wedge \kappa(F) = \eta F \wedge F$ for **any** F.

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- "Convert" wedge products in Levi-Civita symbols. Thus,

$$(\epsilon^{MN} \kappa_M{}^I \kappa_N{}^J) F_I F_J = (\eta \epsilon^{IJ}) F_I F_J \,,$$

for every F_I. Grouping the two terms together, achieve

$$(\epsilon^{MN} \kappa_M{}^I \kappa_N{}^J - \eta \epsilon^{IJ}) F_I F_J = 0 \,.$$

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- Expression in brackets already symmetric under swap of indices $\{I, J\}$. Moreover, it must hold true **for all** F_I, so

$$\epsilon^{MN} \kappa_M{}^I \kappa_N{}^J - \eta \epsilon^{IJ} = 0 \,.$$

Contract by $\hat{\epsilon}_{LI}$ and recall $\hat{\epsilon}_{LI} \epsilon^{IJ} = \delta_L^I$ (Kronecker delta):

$$(\hat{\epsilon}_{LI} \kappa_M{}^I \epsilon^{MN}) \kappa_N{}^J = \eta \delta_L^J \,, \quad \Rightarrow \quad \bar{\kappa}_L{}^N \kappa_N{}^J = \eta \delta_L^J \,.$$

From $I_3 = \eta I_2$ ($\forall$ fields) to mixed closure relation.

- Demand $H \wedge H = \eta F \wedge F$ for **every** choice of H and F. Local & linear media: $\kappa(F) \wedge \kappa(F) = \eta F \wedge F$ for **any** F.
- "Convert" wedge products in Levi-Civita symbols. Thus,

$$(\epsilon^{MN} \kappa_M{}^I \kappa_N{}^J) F_I F_J = (\eta \epsilon^{IJ}) F_I F_J \,,$$

  for every F_I. Grouping the two terms together, achieve

$$(\epsilon^{MN} \kappa_M{}^I \kappa_N{}^J - \eta \epsilon^{IJ}) F_I F_J = 0 \,.$$

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- **Conclude**: imposing $I_3 = \eta I_2$ for all field configurations, leads to the **mixed** closure relation $\bar{\kappa} \circ \kappa = \frac{1}{6} \mathrm{tr}(\bar{\kappa} \circ \kappa) \mathrm{Id}$.

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Preview: the closure relation for skewon-free media.

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2. Solve a **mixed** closure relation. Then, remove skewon. Good: mixed closures easier to solve. They are **useful**.

Closure relations and their properties

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Closure relations at a glance

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- If κ is a **solution** of one closure relation, the respective factor in red vanishes if and only if $\det(\kappa) = 0$. In fact,

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- **If:** κ obeys one pure closure relation;
  Then: $\bar{\kappa}$ satisfies the other pure closure relation.
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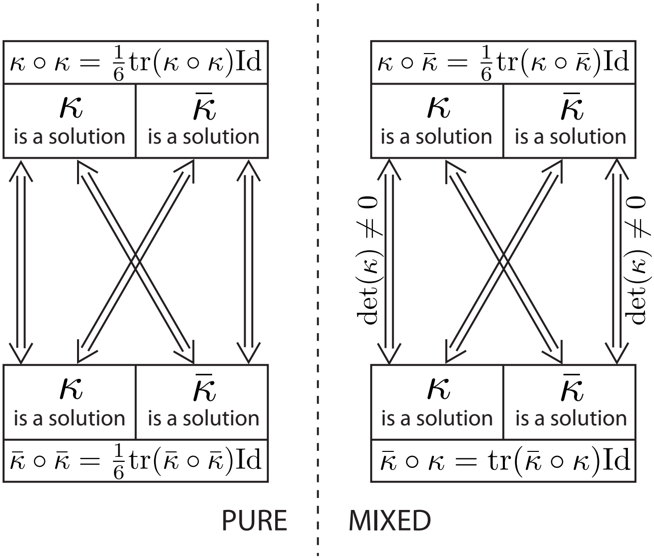
Mixed closure relations

- **If:** κ obeys one mixed closure relation;
  Then: $\bar{\kappa}$ satisfies the other mixed closure relation.
- **If:** κ obeys one mixed closure relation & κ is **invertible**.
  Then: κ satisfies the other mixed closure relation.
  So: the mixed closures have the **same** set of **invertible** solutions. (Not all solutions with $\det(\kappa) = 0$ are shared.)

Links between closure relations



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- Q-media are non-birefringent ($\sim$ Hodge star, vacuum).

Find all invertible roots of mixed closure relations

A. Favaro, L. Bergamin, I.V. Lindell, Y.N. Obukhov.

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Choose to solve $\bar{\kappa} \circ \kappa = \frac{1}{6}\mathrm{tr}(\bar{\kappa} \circ \kappa)\mathrm{Id}$. In components, write:

$$\frac{1}{2}\bar{\kappa}_{\alpha\beta}{}^{\rho\sigma}\kappa_{\rho\sigma}{}^{\mu\nu} = \frac{1}{8}\hat{\epsilon}_{\alpha\beta\gamma\delta}(\kappa_{\eta\theta}{}^{\gamma\delta})\epsilon^{\eta\theta\rho\sigma}\kappa_{\rho\sigma}{}^{\mu\nu} = \frac{1}{6}\mathrm{tr}(\bar{\kappa} \circ \kappa)\delta^{\mu\nu}_{\alpha\beta}$$

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First step: contract expression through by $\frac{1}{2}\epsilon^{\lambda\tau\alpha\beta}$, to obtain

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Using generalised Kronecker delta $\delta^{\eta\theta\rho\sigma}_{\alpha\beta\gamma\delta} = \hat{\epsilon}_{\alpha\beta\gamma\delta}\epsilon^{\eta\theta\rho\sigma}$, yields

$$\frac{1}{4}\delta^{\eta\theta\rho\sigma}_{\alpha\beta\gamma\delta}\kappa_{\eta\theta}{}^{\lambda\tau}\kappa_{\rho\sigma}{}^{\mu\nu} = \frac{1}{6}\mathrm{tr}(\bar{\kappa} \circ \kappa)\epsilon^{\lambda\tau\mu\nu}\hat{\epsilon}_{\alpha\beta\gamma\delta}.$$

Invertible roots of mixed closures (continued)

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The indices in blue and red are made **implicit** by defining the twisted bivector-valued 2-form $\kappa^{\mu\nu}$, and the 4-form density $\hat{\epsilon}$:

$$\kappa^{\mu\nu} := \frac{1}{2!}\kappa_{\alpha\beta}{}^{\mu\nu}(\vartheta^\alpha \wedge \vartheta^\beta),$$
$$\hat{\epsilon} := \frac{1}{4!}\hat{\epsilon}_{\alpha\beta\gamma\delta}(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta).$$

where $\{\vartheta^\alpha\}$ is the **co-frame**. Indeed, by means of $\kappa^{\mu\nu}$ and $\hat{\epsilon}$,

$$(\kappa^{\lambda\tau} \wedge \kappa^{\mu\nu}) = \frac{1}{6}\mathrm{tr}(\bar{\kappa}\circ\kappa)\epsilon^{\lambda\tau\mu\nu}\hat{\epsilon}$$

Implement 6-dimensional indices $\{I, J, \dots\}$, obtain equation

$$(\kappa^I \wedge \kappa^J) = \frac{1}{6}\mathrm{tr}(\bar{\kappa}\circ\kappa)\epsilon^{IJ}\hat{\epsilon}\,,$$

Represent ϵ^{IJ} as a 6×6 matrix formed of four 3×3 **blocks**:

- The diagonal blocks are null matrices $\mathbb{O}_{3\times 3}$.
- The off-diagonal blocks are unit matrices $\mathbb{I}_{3\times 3}$.

Invertible roots of mixed closures (continued)

$$(\kappa^I \wedge \kappa^J) = \frac{1}{6}\mathrm{tr}(\bar{\kappa} \circ \kappa) \left[\begin{array}{c|c} \mathbb{O}_{3\times 3} & \mathbb{I}_{3\times 3} \\ \hline \mathbb{I}_{3\times 3} & \mathbb{O}_{3\times 3} \end{array} \right] \hat{\epsilon} \ ,$$

The **diagonal** of the matrix ϵ^{IJ} is all formed of zeroes, and so

$$\begin{array}{ccc} \kappa^1 \wedge \kappa^1 = 0, & \kappa^2 \wedge \kappa^2 = 0, & \kappa^3 \wedge \kappa^3 = 0, \\ \kappa^4 \wedge \kappa^4 = 0, & \kappa^5 \wedge \kappa^5 = 0, & \kappa^6 \wedge \kappa^6 = 0, \end{array}$$

i.e. the twisted 2-forms $\{\kappa^{\mu\nu}\} = \{\kappa^{01}, \kappa^{02}, \kappa^{03}, \kappa^{23}, \kappa^{31}, \kappa^{12}\}$ must be simple ($\Psi = \alpha \wedge \beta$).

Invertible roots of mixed closures (continued)

$$(\kappa^I \wedge \kappa^J) = \frac{1}{6}\mathrm{tr}(\bar{\kappa} \circ \kappa) \left[\begin{array}{c|c} \mathbb{O}_{3\times 3} & \mathbb{I}_{3\times 3} \\ \hline \mathbb{I}_{3\times 3} & \mathbb{O}_{3\times 3} \end{array} \right] \hat{\epsilon} \ ,$$

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$$\begin{array}{lll} \kappa^1 \wedge \kappa^1 = 0, & \kappa^2 \wedge \kappa^2 = 0, & \kappa^3 \wedge \kappa^3 = 0, \\ \kappa^4 \wedge \kappa^4 = 0, & \kappa^5 \wedge \kappa^5 = 0, & \kappa^6 \wedge \kappa^6 = 0, \end{array}$$

i.e. the twisted 2-forms $\{\kappa^{\mu\nu}\} = \{\kappa^{01}, \kappa^{02}, \kappa^{03}, \kappa^{23}, \kappa^{31}, \kappa^{12}\}$ must be simple ($\Psi = \alpha \wedge \beta$). In fact, a closer analysis of the equation to solve indicates that there are 2 alternative **cases**

A) $\begin{cases} \{\kappa^{01}, \kappa^{02}, \kappa^{03}\} \text{ are simple and all share the same 1-form,} \\ \{\kappa^{23}, \kappa^{31}, \kappa^{12}\} \text{ are simple and pairwise share a different 1-form,} \end{cases}$

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The cases A) & B) respectively correspond to the structures

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- $\mathrm{tr}(\bar{\kappa} \circ \kappa) \equiv \mathrm{tr}(\kappa \circ \bar{\kappa}) = \mathfrak{X}^2(\det Q)$. Consistently with the above, $(\det Q)$ can take **any sign**, but it cannot vanish.

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Invertible solutions to the closure with no skewon.

- Solutions of the P-medium type:

P-medium	$P_\alpha{}^\beta$	Defining property	det P
$\kappa_{\alpha\beta}{}^{\mu\nu} = YL^2\delta^{\mu\nu}_{\alpha\beta}$	$P_\alpha{}^\beta = L\delta^\beta_\alpha$	δ^ρ_α is the identity tensor	$L^4 > 0$
$\kappa_{\alpha\beta}{}^{\mu\nu} = 2YL^2\psi_{[\alpha}{}^\mu\psi_{\beta]}{}^\nu$	$P_\alpha{}^\beta = L\psi_\alpha{}^\beta$	$\psi_\alpha{}^\rho\psi_\rho{}^\beta = \delta^\beta_\alpha$, $\psi_\gamma{}^\gamma = 0$	$L^4 > 0$
$\kappa_{\alpha\beta}{}^{\mu\nu} = 2YM^2J_{[\alpha}{}^\mu J_{\beta]}{}^\nu$	$P_\alpha{}^\beta = MJ_\alpha{}^\beta$	$J_\alpha{}^\rho J_\rho{}^\beta = -\delta^\beta_\alpha$	$M^4 > 0$

$$\mathrm{tr}(\kappa\circ\kappa)\equiv\mathrm{tr}(\bar\kappa\circ\bar\kappa)=\mathrm{tr}(\kappa\circ\bar\kappa)\equiv\mathrm{tr}(\bar\kappa\circ\kappa) = Y^2(\det P).$$

- Solutions of the Q-medium type ($^{[s]}Q^{\alpha\beta}$ is symmetric, while $^{[a]}Q^{\alpha\beta}$ is antisymmetric):

Constitutive relation		det Q
$\kappa_{\alpha\beta}{}^{\mu\nu} = \Omega^{-1}\lvert\det{}^{[s]}Q\rvert^{-\frac{1}{2}}\hat\epsilon_{\alpha\beta\rho\sigma}{}^{[s]}Q^{\rho\mu\,[s]}Q^{\sigma\nu}$	Signature$(^{[s]}Q) = (3,1)$	< 0
$\kappa_{\alpha\beta}{}^{\mu\nu} = \Omega^{-1}(\det{}^{[s]}Q)^{-\frac{1}{2}}\hat\epsilon_{\alpha\beta\rho\sigma}{}^{[s]}Q^{\rho\mu\,[s]}Q^{\sigma\nu}$	Signature$(^{[s]}Q) = (2,2)$	> 0
$\kappa_{\alpha\beta}{}^{\mu\nu} = \Omega^{-1}(\det{}^{[s]}Q)^{-\frac{1}{2}}\hat\epsilon_{\alpha\beta\rho\sigma}{}^{[s]}Q^{\rho\mu\,[s]}Q^{\sigma\nu}$	Signature$(^{[s]}Q) = (4,0)$	> 0
$\kappa_{\alpha\beta}{}^{\mu\nu} = \Upsilon^{-1}(\det{}^{[a]}Q)^{-\frac{1}{2}}\hat\epsilon_{\alpha\beta\rho\sigma}{}^{[a]}Q^{\rho\mu\,[a]}Q^{\sigma\nu}$		> 0

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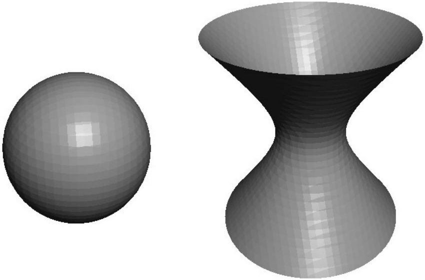
Hodge star based on metric of signature $(3,1)$ easily picked out. More in general, analyse first 3 entries Q-medium table.

Hodge star, analyse various signatures

$$\kappa_{\alpha\beta}{}^{\mu\nu} = \Omega^{-1} |\det{}^{[s]}Q|^{-\frac{1}{2}} \hat{\epsilon}_{\alpha\beta\rho\sigma}{}^{[s]}Q^{\rho\mu\,[s]}Q^{\sigma\nu}$$

- Signature$({}^{[s]}Q) = (3,1)$: Fresnel surface is spherical.
  - $\Omega > 0$: vacuum or medium with scalar positive ϵ, μ.
  - $\Omega < 0$: medium with scalar negative ϵ, μ.
- Signature$({}^{[s]}Q) = (4,0)$: Have only evanescent waves.
  - $\Omega > 0$: metal, plasma, metamaterial (metal rods array).
  - $\Omega < 0$: Metamaterial formed by an array of split rings.
- Signature$({}^{[s]}Q) = (2,2)$: Fresnel surface hyperboloid.
  - $\Omega > 0$: exploited in hyperlens proposed by Jacob, 2006.



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- ◇ Retrieved result concerning Hodge dual, metric (3,1).

Thank-you!