Physical dimensions and units, universal constants and their relativistic invariance

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"Fundamental Physics in Space" Bremen, on 23 to 27 Oct. 2017

I'd like to thank Claus Lämmerzahl for the invitation. Many thanks to Yuri Obukhov and Yakov Itin for joint work on dim. analysis, i.a.

- Post, Formal Structure of Electromagnetics (1962 and Dover 1997).
- H., Itin, Obukhov, Int. J. Theor. Phys. D **25**, No.11 (2016) 1640016.
- Lämmerzahl and H., in preparation (2017/8). *file Bremen2017_03.tex*

Engraving (1575) of 16 citizens of Frankfurt/Main, who define a foot as length scale

String theory at its best (G. Veneziano, 2002): ...
"... it looks unnecessary (and even "silly" according to the present understanding of physical phenomena) to introduce a separate unit for temperature, for electric current and resistance, etc..."

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1. Physical dimensions in class. mechanics

Length ℓ, (angle,) area A, volume V: fundamental and derived dimensions. Dimension of ℓ is $[\ell]$. Compare length of segment 1 with segment 2:

$$\{L_2\}_{cm}/\{L_1\}_{cm} = \{L_2\}_{inch}/\{L_1\}_{inch}$$

Ratio of two length is invariant under change of units. Length is additive. $[\text{area}] = [\text{length}]^2$, derived dimension, $[\text{volume}] = [\text{length}]^3$.

Time t. Still time and length fundamental dim. in SI; what is fundamental is, to a degree, a matter of taste and depends also on the state of knowledge.
Velocity $\mathbf{v}$, with $[v] = [\ell]/[t]$, and acceleration $\mathbf{a}$, with with $[a] = [\ell]/[t]^2$, are both derived dimensions.

mass m, is defined as “quantity of mass”.

force f, measured with a spring scale, e.g..

On the basis is Newton’s equation of motion $\longrightarrow$ interrelates length ℓ , time t, mass m, and force f. Independent are, for instance, (ℓ, t, m).

2. Lagrange-Hamilton formalism, action as a new dimension

The scalar action function with the dimension of action $\mathfrak{h}$ surfaces as a new type of dimension. Following Post (1962,1980), we can opt in classical mechanics alternatively for $(l, t, \mathfrak{h})$ as a basis set of dimensions. $\mathfrak{h}$ is relativistically invariant also in special relativity (SR) and general relativity (GR). Thus, this set is more stable under relativistic conditions (high velocities etc.) than the set (ℓ, t, m).

The mass was experimentally determined by Lavoisiér to be conserved, however, in SR and GR alike, this rule is broken. The explosion of Alamogordo (1945) was a more than clear demonstration of the energy mass equivalence and thus of the non-conservation of mass. Note (in the parentheses we us SI units),

$$\begin{aligned}
\text{action } \mathfrak{h} &= \text{energy} \times \text{time} \quad (Js) \\
&= \text{momentum} \times \text{length} \quad (kg \times ms^{-1} \times m) \\
&= \text{angular momentum} \quad (kg \times ms^{-1} \times m) \; \{\rightarrow Bohr\} \\
&= \text{charge} \times \text{magnetic flux} \quad (C \times Wb = As \times Vs = Js)\,.
\end{aligned}$$

Similarly, momentum flux density = pressure (see momentum flux density of the elmg. field = Maxwell stress)

3. Physical quantity, quantity equations

Dimens. analysis can lead to a better understanding of the *structure* of a phys. theory, see Wallot, Größengleichungen...Leipzig 1957.

$$\begin{aligned} \text{Physical quantity} &\sim \text{some numerical value} \times \text{unit} \\ Q &= \{Q\} \times [Q]. \end{aligned}$$

Example: $T = 23\ h = 23 \times 60\ min = \ldots$ Inverse proportionality rule:

$$Q = \{Q\}' \times [Q]' = \{Q\}'' \times [Q]'' \quad \text{or} \quad \{Q\}'/\{Q\}'' = [Q]''/[Q]'.$$

Physical quantity is invariant w.r.t. the choice of units. For the Π-theorem (Buckingham et al.), see the literature.

The set of all possible units may be called the *dimension* of a quantity, here $[T]$ = time. It is a *qualitative* aspect of a physical quantity. A physical dimension encodes the knowledge of how to set up an experiment to *measure* the quantity.

Equations in physics should form “quantity eqs.” valid for all units ⇒ quantity calculus. “Numerical value eqs.”, used mainly by particle physicists ($c = 1, \hbar = 1, \kappa_{\rm grav} = 1$), are only valid for certain units; usually insight is lost into the physical structure of the corr. theory.

4. Metrology in electromagnetism, a new dimension

Gauss... Weber... Maxwell... Heaviside... Helmholtz... Hertz... Planck... Giorgi... Wallot... $\rightarrow$ SI (International System of Units)

• Gauss and Weber recognized the need for precise measurements in electromagnetism, see also the Weber-Kohlrausch experiment.
• Maxwell recognized the need for a physical quantity as part of the formulas in physics. In the 19th century (see also Planck 1899), electromagnetic units were supposed to be reduced to *mechanical* measurements (Gauss units). This prejudice even propagated into the 21 century to 'modern' textbooks (Jackson).

• Already Giorgi cut the Gordian knot: The need for an independent electrical dimension, the electric resistance Ω, for instance. Later the electric charge q (C) and eventually the el. current j (A) were taken. Nowadays, we can even count single electrons with nano-technical tools. Basic set for dimensional analysis is now $(l, t, \mathfrak{h}, q)$, Post 1962.

Mainly Wallot developed the Maxwellian idea of quantity equations, cf. also Schouten (1954). All of this led to the modern SI ($\sim$1960).

5. Premetric electrodynamics: electric charge & magnetic flux

The premetric Maxwell equations with the Maxwell-Lorentz spacetime relation for vacuum ($\boldsymbol{\partial}\times \to$ curl, $\boldsymbol{\partial}\cdot \to$ div, $\mathfrak{g} = \sqrt{-\det g}\, g$, $g =$ metric)

Physics law	Math. expression
Ampère-Maxwell law	$\boldsymbol{\partial} \times \boldsymbol{\mathcal{H}} - \dot{\boldsymbol{\mathcal{D}}} = \boldsymbol{j}$
Coulomb-Gauss law	$\boldsymbol{\partial} \cdot \boldsymbol{\mathcal{D}} = \rho$
Faraday induction law	$\boldsymbol{\partial} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0$
conserved magnetic flux	$\boldsymbol{\partial} \cdot \boldsymbol{B} = 0$
permittivity of vacuum	$\boldsymbol{\mathcal{D}} = \varepsilon_0 \; \mathfrak{g} \; \boldsymbol{E}$
permeability of vacuum	$\boldsymbol{\mathcal{H}} = \mu_0^{-1} \, \mathfrak{g}^{-1} \, \boldsymbol{B}$

The absolute dimensions of the 4d excitation $G = (\mathcal{D}, \mathcal{H})$ and the 4d field strength $F = (E, B)$ led us to the 4d scalars *electric charge* q and *magnetic flux* ϕ. Their quotient is an *admittance* with dimension $[\lambda_0] = q/\phi$ and their product an *action* with dimension $\mathfrak{h} = q\phi$. In SI, we have, respectively, C, Wb, $1/\Omega$, and Js. These notions, as well as their SI expressions, are true 4d (diffeomorphism invariant) scalars.

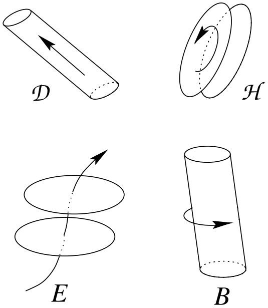
6. Absolute and relative dimensions

Absolute dimensions are assigned to a physical quantity, relative dimensions are those of the components of this quantity with respect to a local tetrad ϑ^α (coframe), with the "legs" $(\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3)$. $[\vartheta^0] =$ time, $[\vartheta^a] =$ length, for $a = 1, 2, 3$.

Mechanics: The 4-momentum $\mathbf{p} = p_\alpha \vartheta^\alpha = p_0\vartheta^0 + p_1\vartheta^1 + p_2\vartheta^2 + p_3\vartheta^3$ of a particle with mass m and velocity $\mathbf{v}$. Assume for $\mathbf{p}$ the (absolute) dimension of an action $\mathfrak{h}$. Then, relative dimensions (of the components of $\mathbf{p}$) $[p_0] = [\mathfrak{h}/t] =$ energy and $[p_a] = [\mathfrak{h}/\ell] = [m\,v] =$ 3d momentum, q.e.d.. Action is a 4d scalar (see Lagrangian formalism).

Lorentz force density and the Maxwell equations: $\mathbf{f} = \rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B}$ and

$dG = J$:	$\operatorname{div}\mathbf{D} = \rho\,, \quad \operatorname{curl}\mathbf{H} - \dot{\mathbf{D}} = \mathbf{j}\,,$
$dF = 0$:	$\operatorname{div}\mathbf{B} = 0\,, \quad \operatorname{curl}\mathbf{E} + \dot{\mathbf{B}} = 0\,.$
4-d Maxwell eqs.	1+3 decomposed Maxwell equations



Faraday–Schouten pictograms of the electromagnetic field

4d *excitation*: $G = (D, H) = D - H \wedge dt \quad \Rightarrow$ coulomb

4d *field strength*: $F = (E, B) = B + E \wedge dt \quad \Rightarrow$ weber

$[G] = q$ (*electric charge*, in SI: coulomb = C) $\Rightarrow$

$$[D] = q,\ [D_{ab}] = \frac{q}{\ell^2} \overset{\text{SI}}{=} \frac{C}{m^2}, \quad [H] = \frac{q}{t},\ [H_a] = \frac{q}{t\ell} \overset{\text{SI}}{=} \frac{C}{sm} = \frac{A}{m}$$

$[F] = \Phi$ (*magnetic flux*, in SI: weber = Wb) $\Rightarrow$

$$[E] = \frac{\Phi}{t},\ [E_a] = \frac{\Phi}{t\,\ell} \overset{\text{SI}}{=} \frac{Wb}{sm} = \frac{V}{m}, \quad [B] = \Phi,\ [B_{ab}] = \frac{\Phi}{\ell^2} \overset{\text{SI}}{=} \frac{Wb}{m^2} = \frac{Vs}{m^2} = T$$

7. Constitutive law relating excitation (D, H) to field strength (E, B)

For local and linear **matter**:

$$\mathfrak{G}^{\lambda\nu} = \frac{1}{2}\,\chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa}\,, \quad \text{with} \quad \chi^{\lambda\nu\sigma\kappa} = -\chi^{\lambda\nu\kappa\sigma} = -\chi^{\nu\lambda\sigma\kappa}\,,$$

where $\chi^{\lambda\nu\sigma\kappa}$ is a *constitutive tensor density* of rank 4 and weight $+1$, with the dimension $[\chi] = [G]/[F] = 1/\mathrm{resistance}$, with 36 independent components.

For **vacuum**: $G = \lambda_0 \, {}^\star F$ (the star means to take the dual). Characteristic *admittance of the vacuum:*

$$[\lambda_0] = \frac{[G]}{[F]} = \frac{q}{\mathfrak{h}/q} = \frac{q^2}{\mathfrak{h}} \overset{\mathrm{SI}}{=} \frac{A^2 s^2}{VAs^2} = \frac{A}{V} = \frac{1}{\Omega} \overset{\text{vacuum}}{\longrightarrow} \frac{1}{377\ \Omega}\,.$$

Is a 4d-scalar. 3d scalars el. const. ε_0 and magn. const. μ_0: $\mathbf{D} = \varepsilon_0 \mathbf{E}$, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$ with $\lambda_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}}$, $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ (speed of light).
Dimensional analysis yields *2 characteristic quantities for vacuum electrodyn.,* the 4d scalars λ_0 and c. Is a result of dimensional analysis, units irrelevant. Also in vacuum $[D] \neq [E]$ and $[H] \neq [B]$.

8.True scalar quantities in terms of q (charge) and $\mathfrak{h}$ (action)

Dimensions $(\mathfrak{h}, q, \ell, t)$, in SI $(m, I, \{\ell, \} t)$. Now, after identifying charge and magnetic flux as fundamental quantities, we can choose alternatively $(q, \Phi, \ell, t) \overset{\text{SI}}{=} (C, Wb, m, s)$ or $(q, \mathfrak{h}, \ell, t) \overset{\text{SI}}{=} (C, Js, m, s)$.
Note that $[q][\Phi] = [\mathfrak{h}]$.
Fundamental quantities should be expressed in

$$q^{n_1}\mathfrak{h}^{n_2} = \text{ 4d scalars} \qquad (n_1, n_2 \text{ numbers}).$$

$$q \to \text{el. charge}, \ \frac{\mathfrak{h}}{q} \to \text{mg. flux}, \ \frac{\mathfrak{h}}{q^2} \to \text{el. resistance}, \ldots$$

We observe $n_1 = \pm 1, -2$; $n_2 = 0, 1$. This is the end of class. eldyn..

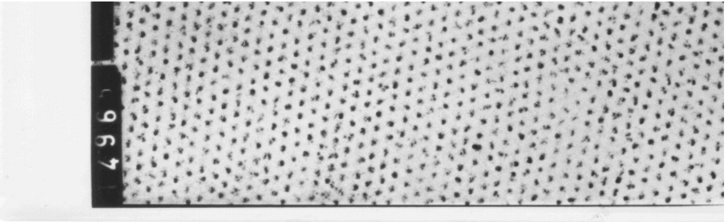
In nature, el. charge is quantized $\to$ elementary charge

$$e \overset{\text{SI}}{\approx} 1.6 \times 10^{-19}\, C \qquad \text{(4d scalar)},$$

and magn. flux can be quantized (in supercond.) $\to$ magnetic fluxon

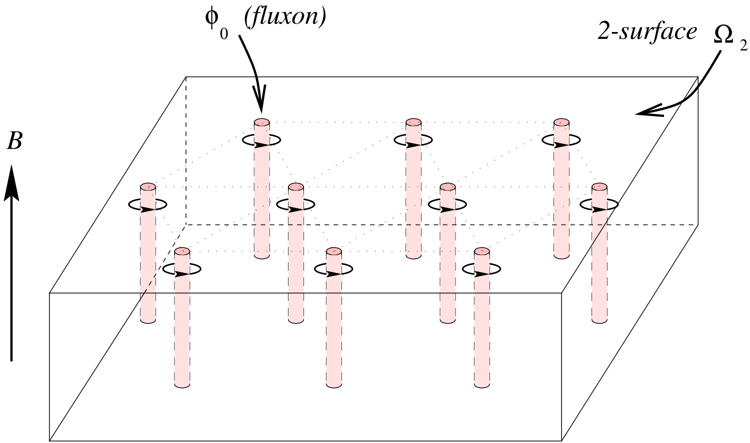
$$\Phi_0 = h/(2e) \overset{\text{SI}}{\approx} 2 \times 10^{-15}\, Wb \quad \text{(4d scalar)}, \ h = \text{ Planck's constant}$$

$2e$ because of the Cooper pair.



Flux lines in type II superconductors according to Essmann & Träuble (1967) (original by courtesy of U.Essmann): Niobium disc (diameter 4 mm, thickness 1 mm), 1.2 K, $\mathcal{H} = 78\,\mathrm{kA/m}$. Parameter of flux line lattice 170 nm.

$$\Phi_0 := \frac{h}{2e} \overset{\mathrm{SI}}{\approx} 2.068 \times 10^{-15}\,\mathrm{weber}$$



Sketch of an Abrikosov lattice in a type II superconductor in 3-dimensional space.

9.Fleischmann: Two classes of universal constants ⇒ *true scalars* and *P-scalars*

Fleischmann (1971): In physics there are on one side 4d laws, which do not contain the metric g_{ik} and are covariant under general coordinate transformations (diffeomorphism covariant), on the other side those 4d laws, in which the metric g_{ik} is involved. To the former belong the Maxwell equations, to the latter their constitutive relations.

In the case of scalars: ⇒ true 4d scalars, ⇒ 4d P(oincaré)-scalars. Poincaré (inhomogeneous Lorentz) group is the group of motions of flat Minkowski space (no gravity!)

True 4d scalars: Only a few, namely action S, electric charge Q, magnetic flux Φ, entropy K, as well as products and quotients thereof. Interestingly enough, in nature, all of those true 4d scalars are changed via quanta. Note, Hamiltonians, energies, energy densities, and masses, e.g., are not 4d scalars.

4d P-scalars: We know only of the speed of light c.

10. Josephson constant K_J and von Klitzing constant R_K as true 4d scalars

Pick for q the elementary charge e and for $\mathfrak{h}$ the Planck constant h: arrive at the *Josephson* and the *von Klitzing* constants of modern metrology, which provide highly precise measurements of e and h (peta=P=10^{15}):

$$K_{\rm J} = \frac{2e}{h} \overset{\rm SI}{\approx} 0.483\,\frac{\rm PHz}{\rm V} \approx \frac{1}{2.068\times 10^{-15}{\rm Wb}}\,,\quad R_{\rm K} = \frac{h}{e^2} \overset{\rm SI}{\approx} 25.813\,{\rm k\Omega}$$

We know that K_J and R_K are true 4d scalars, since they have the dimension of a reciprocal magnetic flux and a resistance, respectively. Incidentally, by the same token, the Quantum Hall Effect is not influenced by the grav. field (H., Obukhov, Rosenow: PRL 2004).

The SI of post-2018, will be built on the true 4d scalars K_J and R_K, since the Josephson and the von Klitzing (QHE) effects belong to the most precise tools in metrology (Mössbauer related to a frequency).

Note $\sqrt{\varepsilon_0/\mu_0} = e^2/(2h\alpha), \quad \alpha = e^2/(2\varepsilon_0\, c\, {\rm h}) = \Omega_0/(2R_{\rm K}),$

with α as fine struct. const.. All these quantities are true 4d scalars.

11. Can we measure the speed of light in a gravitational field?

1. SR is based (i) on the special relativity principle (equivalence of all inertial frames) and (ii) on the principle of the constancy of the speed of light c in vacuo. Thus, by construction, c is a P-scalar (Fleischmann class 2). The elmg. vacuum constitutive law for isotropic light propagation in components: $\mathfrak{G}^{\lambda\nu} = \frac{1}{2}\,\chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa}$ with $\chi^{\lambda\nu\sigma\kappa} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{-g}\left(g^{\lambda\sigma}g^{\nu\kappa} - g^{\nu\sigma}g^{\lambda\kappa}\right)$. The metric is vital in this law.

2. GR: A gravitating body (star, Sun) deflects light. Gravity acts like a refractive medium. Speed of light depends on gravity: $c = c(g)$. Shapiro effect (delay of radar echos at inner planets). Constancy of the speed of light is invalidated. In a freely falling laboratory (ISS), you need 2 points, a finite distance apart, for measuring c, geodesic deviation (and thus the curvature of spacetime) cannot be neglected.

3. In GR conventionally a c_0 is defined (opening of the light cone in GR), with $c_0 \neq c(g)$. For c_0, the numerical value of SR is adopted. However, c_0 cannot be realized as measurement of the speed of light in the grav. field. By convention (!), c_0 is assumed to be a true 4d scalar (thanks to Volker Perlick, Bremen, for discussions).

4. In metrology in the SI community, the unit of length is reduced to the unit of time by ruling that c has a certain numerical value, is a conversion factor. But if c is only a 4d P-scalar, this rule seems self-contradictory. By definition, c is a P-scalar, is a P-scalar... I don't know literature, which did address this problem. Has this problem not been brought up, because the effects emerging would be minute?

A recent paper discusses the precision with which the speed of light can be measured: Braun, Schneiter, Fischer, "Intrinsic measurement errors for the speed of light in vacuum," CQG **34**, 175009 (2017). But therein mainly quantum field theoretical corrections are determined. Tacitly, they assume c as a true 4d scalar.

5. Recently the event GW170817 was observed gravitationally and electromagnetically likewise. Perhaps the delay time between these measurements elucidate the problem c_0 is really a true 4d scalar. See, e.g., Wei *et al.*, *Multimessenger tests of the weak equivalence principle from GW170817 and its electromagnetic counterparts,* arXiv:1710.05860, and
Shoemaker, Murase, *Constraints from the Time Lag between Gravitational Waves and Gamma Rays: Implications of GW 170817 and GRB 170817A,* arXiv:1710.06427.

C.C.A.A. = Colonia Claudia Ara Agrippinensium = Köln