The constitutive tensor of linear elasticity: its decompositions, Cauchy relations, null Lagrangians, and wave propagation

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Herrn Claus Lämmerzahl sei recht herzlich für die Einladung gedankt!

H., Itin: J. Elasticity 66, 185 (2002); Itin, H.: J. Math. Phys. 54, 042903 (2013); H., Obukhov: Foundations of Classical Electrodynamics: charge, flux, and metric, Birkhäuser, Boston (2003)

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Abstract: We study properties of the fourth rank elasticity tensor C within linear elasticity theory. First C is irreducibly decomposed under the linear group into a "Cauchy piece" S (with 15 independent components) and a "non-Cauchy piece" A (with 6 independent components). Subsequently, we turn to the physically relevant orthogonal group, thereby using the metric. We find the finer decomposition of S into pieces with 9+5+1 and of A into those with 5+1 independent components. Some reducible decompositions, discussed earlier by numerous authors, are shown to be inconsistent. — Several physical consequences are discussed. The Cauchy relations are shown to correspond to A=0. Longitudinal and transverse sound waves are basically related by S and A, respectively.

1.1 Overview

- Local & linear (in general anisotropic) elasticity, important for steel, e.g., in civil and mechanical engineering, valid in linear approaximation.

- stress $\sim$ elasticity tensor $\times$ strain, or $\sigma^{ij} = C^{ijkl}\varepsilon_{ij}$, with $i, j, ... = 1, 2, 3$ ("ut tensio, sic vis", Hooke).

- Both, stress and strain are symmetric, $\sigma^{ij} = \sigma^{ji}$ and $\varepsilon^{ij} = \varepsilon^{ji}$, they carry each 6 independent components.

- The symmetry of σ^{ij} follows from the assumptions of vanishing torque stresses and vanishing body couples (body of classical elasticity, "Boltzmann's axiom"), the symmetry of ε_{ij} is due to ε_{ij} being the difference between two metric tensors.

- Thus, $C^{ijkl} = C^{jikl} = C^{ijlk}$ corresponds to a 6×6 matrix $\mathbf{C}$ (Voigt) with 36 independent components: $11 \to 1, 22 \to 2, 33 \to 3, 23 \to 4, 31 \to 5, 12 \to 6$ or $C^{ijkl} \to \mathbf{C}^{IK}$. It represents the elastic properties of the material involved.

- If the deformations are reversible, the elasticity matrix is symmetric, $\mathbf{C} = \widetilde{\mathbf{C}}$, or $C^{ijkl} = C^{klij}$; then, it has 21 independent components; energy density $\sim \frac{1}{2}\sigma^{ij}\varepsilon_{ij} = \frac{1}{2}C^{ijkl}\varepsilon_{ij}\varepsilon_{kl}$.

- For each crystal class, e.g., $\mathbf{C}$ has a certain number of independent components. Isotropic body: 2 Lamé constants (λ, μ) or (E, ν), elasticity (Young's) modulus and Poisson number.

- Our main goal is to find a suitable decomposition of C^{ijkl} and its physical meaning—and to exclude some incorrect decompositions that are prevalent in the literature.

1.2 Related decomposition in local & linear 4d electrodynamics

▶ Excitation $\mathfrak{G}^{\mu\nu}(\mathbf{D}, \mathbf{H}) = -\mathfrak{G}^{\nu\mu}$ and field strength $F_{\mu\nu}(\mathbf{E}, \mathbf{B}) = -F_{\nu\mu}$, here $\mu, \nu = 0, 1, 2, 3$. Maxwell's eqs. in 4d premetric form

$$\partial_\nu \mathfrak{G}^{\mu\nu} = \mathfrak{J}^\mu, \qquad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0.$$

▶ Constitutive law for local & linear electromagnetic media (elmg. response tensor)

$$\mathfrak{G}^{\lambda\nu} = \frac{1}{2}\chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa}, \qquad \chi^{\lambda\nu\sigma\kappa} = -\chi^{\nu\lambda\sigma\kappa} = -\chi^{\lambda\nu\kappa\sigma}, \qquad \chi \to 6 \times 6 \text{ matrix}$$

(in vacuum, with metric $g^{\alpha\beta}$, find $\mathfrak{G}^{\lambda\nu} = Y_0\sqrt{-g}g^{\lambda[\alpha|}g^{\nu|\beta]}F_{\alpha\beta} = Y_0\sqrt{-g}F^{\lambda\nu}$).

▶ Decompose the constitutive tensor irreducibly, with $[\chi] = Y$ = admittance,

$$\chi^{\lambda\nu\sigma\kappa} = \overbrace{^{(1)}\chi^{\lambda\nu\sigma\kappa}}^{\text{principal part}} + \overbrace{^{(2)}\chi^{\lambda\nu\sigma\kappa}}^{\text{skewon part}} + \overbrace{^{(3)}\chi^{\lambda\nu\sigma\kappa}}^{\text{axion part}}$$

$$36 = 20 \oplus 15 \oplus 1$$

$^{(3)}\chi^{\lambda\nu\sigma\kappa} = \widetilde{\alpha}\,\widetilde{\epsilon}^{\lambda\nu\sigma\kappa}$ is totally antisymmetric with $\widetilde{\epsilon}_{\lambda\nu\sigma\kappa} = \pm 1, 0$.

▶ Reversibility: $^{(2)}\chi^{\lambda\nu\sigma\kappa} = 0$ and $\mathfrak{G}^{\lambda\nu} = \frac{1}{2}\left(^{(1)}\chi^{\lambda\nu\sigma\kappa} + \widetilde{\alpha}\,\widetilde{\epsilon}^{\lambda\nu\sigma\kappa}\right)F_{\sigma\kappa}$. Split it into space and time ($a, b, \cdots = 1, 2, 3$)

$$D^a = \varepsilon^{ab}E_b + \gamma^a{}_b B^b + \widetilde{\alpha}\,B^a,$$
$$H_a = \mu_{ab}^{-1}B^b - \gamma^b{}_a E_b - \widetilde{\alpha}\,E_a.$$

We have the 6 permittivities $\varepsilon^{ab} = \varepsilon^{ba}$, the 6 permeabilities $\mu_{ab} = \mu_{ba}$, and the 8+1 magnetoelectric pieces $\gamma^a{}_b$ (with $\gamma^c{}_c = 0$) and $\widetilde{\alpha}$, respectively; hence, altogether 21 indep. components; $\widetilde{\alpha}$ is a 4d pseudoscalar (the axion part)!

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1.3 Related decomp. in gauge theory of gravity and in general relativity

- ▶ Curvature tensor $R^{\mu\nu\kappa\lambda}$ in a Riemann-Cartan space (Poincaré gauge theory of gravity PG): decomposition by E. Cartan, Wallner, McCrea, and H.. Symmetries:

$$R^{\mu\nu\kappa\lambda} = -R^{\nu\mu\kappa\lambda} = -R^{\mu\nu\lambda\kappa}, \qquad R \to 6 \times 6 \text{ matrix};$$

36 independent components. Standard PG decomposition into

Weyl 10 + Paircom 9 + Pscalar 1 + Ricsymf 9 + Ricanti 6 + scalar 1 → 36.

- ▶ Curvature tensor $R^{\mu\nu\kappa\lambda}$ in a Riemann space (general relativity GR) has an additional symmetry ('reversibility'):

$$R^{\mu\nu\kappa\lambda} = R^{\kappa\lambda\mu\nu}, \qquad R \to \text{ symmetric traceless } 6 \times 6 \text{ matrix};$$

20 independent components. Standard GR decomposition into

Weyl 10 + tracefree Ricci 9 + curvature scalar 1 → 20.

- ▶ Accordingly, the curvature tensor $R^{\mu\nu\kappa\lambda}$ has some similarity with the electromagnetic response tensor $\chi^{\mu\nu\kappa\lambda}$ and the elasticity tensor C^{ijkl}: all correspond to 6×6 matrices of some kind.

2.1 Total recall: elasticity tensor and its symmetries

▶ In local & linear elasticity for a homogeneous anisotropic body, the generalized Hooke law postulates a linear relation between the two second-rank tensor fields, the stress σ^{ij} and the strain ε_{kl}:

$$\boxed{\sigma^{ij} = C^{ijkl}\,\varepsilon_{kl}} \qquad\qquad C^{ijkl} \text{ — elasticity tensor}.$$

▶ Since ε_{ij} and σ^{ij} are symmetric, certain symmetry relations hold also for the elasticity tensor. Thus, it obeys the right minor symmetry and the left minor symmetry, respectively (we have $[ij] := \{ij - ji\}/2$):

$$\varepsilon_{[kl]} = 0 \quad\Longrightarrow\quad \boxed{C^{ij[kl]} = 0} \qquad \text{and} \qquad \sigma^{[ij]} = 0 \quad\Longrightarrow\quad \boxed{C^{[ij]kl} = 0}$$

▶ The energy density of a deformed material is expressed as $W = \frac{1}{2}\,\sigma^{ij}\varepsilon_{ij}$. When the Hooke law is substituted, this expression takes the form

$$W = \tfrac{1}{2}\,C^{ijkl}\varepsilon_{ij}\varepsilon_{kl}\,.$$

The right-hand side involves only those combinations of the elasticity tensor components which have the major symmetry,

$$\boxed{C^{ijkl} = C^{klij}}$$

Therefore, only 21 independent components of C^{ijkl} are left over.

2.2 Reducible MN decomposition of C^{ijkl}

- In the literature on elasticity, a special decomposition of C^{ijkl} into two tensorial parts is frequently used: *Cowin (1992), Campanella & Tonton (1994), Podio-Guidugli ("Primer in Elasticity" 2000), Weiner (2002), Haussühl (2002/07), etc.;* note $(ij) := \{ij + ji\}/2$:

$$\boxed{C^{ijkl} = M^{ijkl} + N^{ijkl}}, \quad \text{with} \quad \boxed{M^{ijkl} := C^{i(jk)l}} \quad \text{and} \quad \boxed{N^{ijkl} := C^{i[jk]l}}.$$

This decomposition induces, according to Cowin et al., important physical consequences

- **Cauchy relations**: they appear in the old 'Cauchy' elasticity model (basically due to de St.Venant) that involves only 15 independent combinations of C^{ikjl}. In recent publications, they are postulated incorrectly as

$$N^{ijkl} = 0 \qquad \text{or} \qquad C^{ijkl} = C^{ikjl}.$$

- **Null Lagrangian**: it is defined as that part of the strain energy functional that does not contribute to the equilibrium equation. The energy density of a deformed material is expressed as

$$W = 2M^{ijkl}\partial_j u_i \partial_k u_l + 2N^{ijkl}\partial_j(u_i \partial_k u_l).$$

The second N-term is a total derivative. This result was interpreted by *Lancia, Vergara Cafarelli, and Podio-Guidugli* incorrectly by claiming that the 6 non-Cauchy elasticty constants are not involved in the dynamical equations.

2.3 Algebraic problems with the MN-decomposition

▶ The major symmetry holds for both tensors M^{ijkl} and N^{ijkl}:

$$M^{[ij]kl} = M^{ij[kl]} = 0 \qquad \text{and} \qquad N^{[ij]kl} = N^{ij[kl]} = 0 \,.$$

But, in general, the minor symmetries do not hold,

$$M^{ijkl} \neq M^{klij} \qquad \text{and} \qquad N^{ijkl} \neq N^{klij} \,.$$

Thus, M and N themselves cannot serve as elasticity tensors for any material.

▶ We find for the vector spaces of the C, M, N-tensors,

$$\dim C = 21 \,, \qquad \dim N = 6 \,, \qquad \text{but} \qquad \dim M = 21 \,.$$

▶ Inconsistency: In general, some components of C^{ijkl} can be expressed in different ways in terms of M^{ijkl} and N^{ijkl}. For instance,

$$C^{1223} = M^{2123} + N^{2123} = M^{2132} \,.$$

▶ Reducibility: Since, in general, the tensor M^{ijkl} is not completely symmetric, a finer decomposition is possible, $M^{ijkl} = M^{(ijkl)} + K^{ijkl}$. Accordingly, C^{ijkl} can be decomposed into three tensorial pieces:

$$C^{ijkl} = M^{(ijkl)} + K^{ijkl} + N^{ijkl} \,.$$

2.4 Unique irreducible decomp. of C^{ijkl} under the linear group $GL(3,R)$

- The decomposition
$$C^{ijkl} = S^{ijkl} + A^{ijkl}$$
with

$$S^{ijkl} := C^{(ijkl)} = \frac{1}{3}(C^{ijkl} + C^{iklj} + C^{iljk}) \qquad \text{and} \qquad A^{ijkl} := C^{ijkl} - S^{ijkl}$$

  is irreducible and unique. The irreducibility of S^{ijkl} is obvious, since under the linear group no traces can be taken.

- The partial tensors satisfy the minor symmetries

$$S^{[ij]kl} = S^{ij[kl]} = 0 \qquad \text{and} \qquad A^{[ij]kl} = A^{ij[kl]} = 0\,.$$

  and the major symmetry

$$S^{ijkl} = S^{klij} \qquad \text{and} \qquad A^{ijkl} = A^{klij}\,.$$

  Thus, S and A can pose—in contrast to M and N—as elasticity tensors.

- The irreducible decomposition of C signifies the reduction of $\mathcal{C}$ to the direct sum of its two subspaces, $\mathcal{S} \subset \mathcal{C}$ for the tensor S, and $\mathcal{A} \subset \mathcal{C}$ for the tensor A,

$$\mathcal{C} = \mathcal{S} \oplus \mathcal{A}, \quad \text{in particular,} \quad \dim C = 21\,, \qquad \dim S = 15\,, \qquad \dim A = 6\,.$$

  For rigorous proofs, using the Young decomposition techniques, see our publications.

If we reduce the $GL(3, R)$ according to $GL(3, R) \to SL(3, R) \to SO(3)$, we find
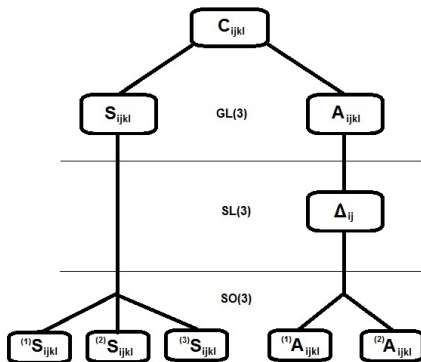
The decomposition tree of the elasticity tensor Cijkl:
Note that Δ_{ij} is a tensor (and not a density) under the $SL(3, R)$.

By taking traces of S and A, we find two symmetric tensors, that is, $5 + 1$ components; for S, the rest amounts to 9 independent components.

3.1 Cauchy relations defined

▶ The auxiliary quantity N^{ijkl} can be expressed in terms of the irreducible elasticity A^{ijkl} as follows:

$$N^{ijkl} = A^{i[jk]l} \qquad \text{or} \qquad A^{ijkl} = \frac{4}{3} N^{ij(kl)}.$$

▶ The irreducible elasticity A^{ijkl} can be equivalently described by a symmetric 2nd rank tensor (this can be seen by counting)

$$\Delta_{mn} := \frac{1}{4} \epsilon_{mik} \epsilon_{njl} A^{ijkl}, \qquad \text{with the inverse} \qquad A^{ijkl} = \frac{4}{3} \epsilon^{i(k|m} \epsilon^{j|l)n} \Delta_{mn}.$$

▶ Cauchy relations are equivalently expressed as

$$A^{ijkl} = 0 \qquad \text{or} \qquad \Delta_{mn} = 0 \qquad [\text{or} \qquad N^{ijkl} = 0.]$$

However, in contrast to the stipulations of Podio-Guidugli, the tensor N^{ijkl} cannot serve as a non-Cauchy measure, because it is not an elasticity tensor. Only when $N^{ijkl} = 0$ is assumed, the tensor M^{ijkl} is restricted to 15 independent components. For most materials the Cauchy relations do not hold even approximately.

▶ In fact, the elasticity of a generic anisotropic material is described by the whole set of the 21 independent components $\mathbf{C}^{IJ}$ and not by a restricted set of 15 independent components obeying the Cauchy relations. This fact seems to nullify the importance of the Cauchy relations for modern solid state theory and leave them only as historical artifact.

This is not true, as you show on the next page. The Cauchy relations provide important hints to the molecular interactions of the building blocks of a crystal.

3.2 Cauchy versus non-Cauchy parts in elasticity

- ► However, a lattice-theoretical approach shows that the Cauchy relations are valid provided:
    1. The interaction forces between the atoms or molecules of a crystal are central forces, as, for instance, in rock salt,
    2. each atom or molecule is a center of symmetry,
    3. the interaction forces between the building blocks of a crystal can be well approximated by a harmonic potential.

- ► Accordingly, a study of the violations of the Cauchy relations yields important information about the intermolecular forces of elastic bodies. One should look for the deviation of the elasticity tensor from its Cauchy part. This deviation measure, being a macroscopic characteristic of the material, delivers important information about the microscopic structure of the material. It must be defined in terms of a unique proper decomposition of the elasticity tensor.

- ► Experimental values for $\Delta_{11} \propto \widetilde{\Delta}_{11} = 3(C_{11} - C_{66})/(C_{11} + 2C_{12})$ are, according to Haussühl (2007),

NaCl	KCl	NaCN	KCN	LiH	Li$_2$O	LiF	BeO	MgO	AgCl
0.010	0.031	0.78	0.72	−0.95	−0.45	−0.26	−0.24	−0.39	0.81

$\widetilde{\Delta}_{11}$ is a quotient of non-Cauchy and Cauchy parts and as such dimensionless, see the literature suggested.

- ► The problem of the identification of the deviation part is solved when the irreducible decomposition is used. In this case, we can define the main or Cauchy part, as given by the tensor S^{ijkl} with 15 independent components, and the deviation or non-Cauchy part, presented by the tensor A^{ijkl} with 6 independent components.

3.3 Does there exist a null Lagrangian in linear elasticity?

- ▶ Are there some elasticity constants or their combinations that do not contribute to the energy and the equation of motion?

- ▶ Lancia, Vergara Cafarelli, and Podio-Guidugli propose:

$$W = 2C^{ijkl}\partial_i u_j \partial_k u_l = 2M^{ijkl}\partial_j u_i \partial_k u_l + 2N^{ijkl}\partial_j(u_i \partial_k u_l).$$

The N-term is a total derivative, thus, only the M-term is involved in the variational principle for deriving the equations of motion. This result was interpreted by Lancia et al. as solving the null Lagrangian problem for linear elasticity theory, since the additional M-tensor is the only one that is involved in the equilibrium equation. However, this statement does not have an invariant meaning. Indeed, the remaining part with the M-tensor has exactly the same set of 21 independent components as the initial C-tensor.

- ▶ Dynamical equations are completely equivalent

$$M^{ijkl}\partial_j \partial_k u_l = 0 \quad \Longleftrightarrow \quad C^{ijkl}\partial_j \partial_k u_l = 0$$

Any total derivative term in the elastic energy functional can be regarded as only a formal expression. It does not remove any subset of the elastic constants from the equilibrium equation, that is, for an arbitrary anisotropic material an elastic null Lagrangian does not exist.

4.1 Equations of motion

- The acoustic wave propagation in anisotropic media is described by the following equation:

$$\rho g^{il} \ddot{u}_l - C^{ijkl} \partial_j \partial_k u_l = 0 \,.$$

- With a plane wave ansatz

$$u_l = U_l e^{i(\zeta n_j x^j - \omega t)} \,,$$

we obtain a system of three homogeneous algebraic equations

$$\left(\rho \omega^2 g^{il} - C^{ijkl} \zeta^2 n_j n_k \right) U_l = 0 \,.$$

- With the definitions of the Christoffel tensor

$$\Gamma^{il} := (1/\rho)\, C^{ijkl} n_j n_k$$

and of the phase velocity $v := \omega/\zeta$, the algebraic equations take the form

$$\left(v^2 g^{il} - \Gamma^{il} \right) U_l = 0 \,,$$

whereas the characteristic equation reads $\boxed{\det \left(v^2 g^{il} - \Gamma^{il} \right) = 0}$.

4.2 Christoffel tensor and its decomposition

► Substituting the irreducible SA-decomposition into Γ^{il}, we obtain

$$\Gamma^{il} = \mathcal{S}^{il} + \mathcal{A}^{il}\,,$$

where

$$\mathcal{S}^{il} := S^{ijkl} n_j n_k = \mathcal{S}^{li}\,, \qquad \text{and} \qquad \mathcal{A}^{il} = A^{ijkl} n_j n_k = \mathcal{A}^{li}\,.$$

► The explicit expressions

$$\mathcal{S}^{il} = \frac{1}{3\rho}(C^{ijkl} + C^{iklj} + C^{iljk}) n_j n_k$$ — Cauchy Christoffel tensor,

and

$$\mathcal{A}^{il} = \frac{1}{3\rho}\left(2C^{ijkl} - C^{iklj} - C^{ilkj}\right) n_j n_k$$ — non-Cauchy Christoffel tensor .

► For each elasticity tensor C^{ijkl} and for each wave covector n_i,

$$\mathcal{A}^{il} n_l = 0 \qquad \text{and} \qquad \det\left(\mathcal{A}^{ij}\right) = 0\,.$$

The latter equation shows why in nature no materials can exist with only a non-Cauchy elasticity tensor. The wave propagation problem in such material is ill-posed.

4.3 Polarization of acoustic waves

▶ Acoustic wave propagation in an elastic medium is an eigenvector problem

$$\left(v^2 g^{il} - \Gamma^{il}\right) U_l = 0\,,$$

in which the phase velocity v^2 are the eigenvalues. In general, three distinct real positive solutions correspond to three independent waves $^{(1)}U_l$, $^{(2)}U_l$, and $^{(3)}U_l$, called acoustic polarizations.

▶ For isotropic materials, there are three pure polarizations:
one longitudinal (or compression) wave with

$$\vec{U} \times \vec{n} = 0\,,$$

and two transverse (or shear) waves with

$$\vec{U} \cdot \vec{n} = 0\,,$$

▶ In general, for anisotropic materials, three pure modes do not exist.
The identification of the pure modes and the condition for their existence is an interesting problem.
Let us see how the irreducible decomposition of the elasticity tensor, which we applied to the Christoffel tensor, can be used here.

Polarization of acoustic waves (cont.)

► For a chosen direction vector $\vec{n}$, we introduce a vector and a scalar according to

$$S^i := \Gamma^{ij} n_j = S^{ij} n_j \,, \qquad S := \Gamma^{ij} n_i n_j = S^{ij} n_i n_j \,.$$

► Let n^i denotes an allowed direction for the propagation of a compression wave. Then the velocity v_{L} of this wave in the direction of n^i is determined only by the Cauchy part of the elasticity tensor:

$$v_{\mathrm{L}} = \sqrt{S} \,.$$

► In which directions can the three pure polarizations propagate? For a medium with a given elasticity tensor, all three purely polarized waves (one longitudinal and two transverse) can propagate in the direction $\vec{n}$ if and only if

$$S^i = S n^i \,.$$

Accordingly, for a given medium, the directions of the purely polarized waves depend on the Cauchy part of the elasticity tensor alone.

► In other words, two materials, with the same Cauchy parts S^{ijkl} of the elasticity tensor but different non-Cauchy parts A^{ijkl}, have the same pure wave propagation directions and the same longitudinal velocity.

4.4 Isotropic example

For clarification, consider an isotropic elastic medium. Then, the elasticity tensor reads

$$C^{ijkl} = \lambda\, g^{ij} g^{kl} + \mu \left(g^{ik} g^{lj} + g^{il} g^{jk} \right) = \lambda\, g^{ij} g^{kl} + 2\mu\, g^{i(k} g^{l)j}\,,$$

with the Lamé moduli λ and μ, see Marsden & Hughes. The irr. decomposition yields

$$S^{ijkl} = (\lambda + 2\mu)\, g^{(ij} g^{kl)} = \alpha \left(g^{ij} g^{kl} + g^{ik} g^{lj} + g^{il} g^{jk} \right)\,,$$

and

$$A^{ijkl} = \beta \left(2g^{ij} g^{kl} - g^{ik} g^{lj} - g^{il} g^{jk} \right)\,.$$

It reveals two fundamental isotropic elasticity constants ($\beta = 0$ leads back to Cauchy with $\lambda = \mu$ and Poisson number $\nu = 1/4$, 'rari-' vs. 'multi-'constant theory)

$$\alpha := \frac{\lambda + 2\mu}{3}\,, \quad \beta := \frac{\lambda - \mu}{3}\,.$$

Then, the characteristic equation for the acoustic waves takes the form

$$\det \left[(v^2 - \alpha + \beta) g^{ij} - (2\alpha + \beta) n^i n^j \right] = 0\,.$$

The longitudinal wave velocity is a solution

$$\boxed{v_1^2 = S = 3\alpha = \lambda - 2\mu\,,}$$

Another solution (of multiplicity 2) with 2 shear waves (μ = shear modulus)

$$\boxed{v_2^2 = \alpha - \beta = \mu\,.}$$

4.5 Anisotropic example

- With the use of the irreducible decomposition, we can answer now the following question:
- Does there exist an anisotropic medium in which purely polarized waves can propagate in every direction?
- It is clear that media with the same Cauchy part of the elasticity tensor (and arbitrary non-Cauchy part) have the same property in this respect. As a consequence we have:
- The most general type of an anisotropic medium that allows propagation of purely polarized waves in an arbitrary direction has an elasticity tensor of the form

$$
C^{ijkl} = \begin{bmatrix}
\alpha & \alpha/3 + 2\rho_1 & \alpha/3 + 2\rho_2 & 2\rho_3 & 0 & 0 \\
* & \alpha & \alpha/3 + 2\rho_4 & 0 & 2\rho_5 & 0 \\
* & * & \alpha & 0 & 0 & 2\rho_6 \\
* & * & * & \alpha/3 - \rho_4 & -\rho_6 & -\rho_5 \\
* & * & * & * & \alpha/3 - \rho_2 & -\rho_3 \\
* & * & * & * & * & \alpha/3 - \rho_1
\end{bmatrix},
$$

where $\rho_1, \cdots, \rho_6$ are arbitrary parameters. The velocity of the longitudinal waves in this medium is $v_{\mathrm{L}} = \sqrt{3\alpha} = \sqrt{\lambda - 2\mu}$.

5. Discussion

A fourth rank tensor of type $\binom{4}{0}$ in three-dimensional Euclidean space qualifies to describe anisotropic elasticity if

1. its physical components carry the dimension of force/area (in SI pascal),
2. it obeys the left and right minor symmetries,
3. it obeys the major symmetry, and
4. it is positive definite.

It is then called elasticity tensor C^{ijkl} (or 'elasticity' or 'stiffness'). Its decomposition into Cauchy and non-Cauchy parts is of fundamental physical importance.
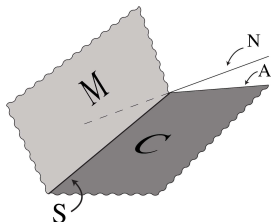
A tensor T^{ijkl} of rank 4 in 3d space has $3^4 = 81$ independent components. The 3d's of our image represent this 81d space. The plane $\mathbb{C}$ depicts the 21 dimensional subspace of all possible elasticity tensors. This space is span by its irreducible pieces, the 15d space of the totally symmetric elasticity $\mathbf{S}$ (a straight line) and the 6d space of the difference $A = \mathbb{C} - \mathbf{S}$ (also depicted, we are sad to say, as a straight line).— Oblique to $\mathbb{C}$ is the 21d space $\mathbb{M}$ of the reducible M-tensor and the 6d space of the reducible N-tensor. The $\mathbb{C}$ "plane" is the only place where elasticities are at home. The spaces $\mathbb{M}$ and N represent only elasticities, provided the Cauchy relations are fulfilled. Then, $A = N = 0$ and $\mathbb{M}$ and $\mathbb{C}$ cut in the 15d space of $\mathbf{S}$. The spaces M and C are intersecting exactly on S.