Nonlocal Gravity Simulates Dark Matter

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I'd like to thank Manuel Hohmann and his co-organizers for giving me the possibility to present a classical nonlocal theory of gravity.

Thanks for joint work on nonlocal gravity to Bahram Mashhoon and on gauge theories of gravity to Yuri Obukhov, Yakov Itin, and Jens Boos.

- FWH, B. Mashhoon, PLB **673**, 279 (2009), PRD **79**, 064028 (2009)
- B. Mashhoon, *Nonlocal Gravity,* Oxford 2017
- Itin, H, Obukhov, PRD **95,** 084020 (2017)

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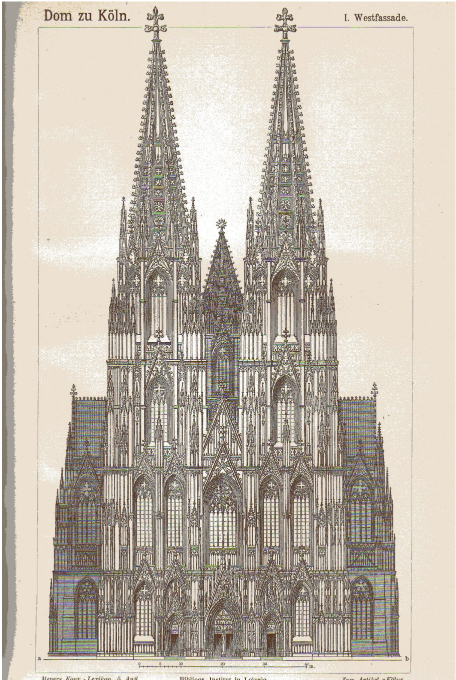
C.C.A.A. = Colonia Claudia Ara Agrippinensium = Köln



Contents

1. Postulate of locality and its limits

- In SR, generalize Poincaré transformations from inertial observers to accelerated observers. For a Newtonian point particle, the initial configuration of which is specified by position and velocity, this is obvious (point coincidences):
- Postulate of locality: An accelerated observer (measuring device) along its *worldline* is at each instant physically equivalent to a hypothetical inertial observer (measuring device) that is otherwise identical and instantaneously *comoving* with the accelerated observer (measuring device).
- Acceleration lengths for an Earth-bound laboratory for translational acceleration $\ell_{\text{tr}} := c^2/g_\oplus \approx 1$ light year $\approx 10^{16}$ m, for rotat. accel. $\ell_{\text{rot}} := c/\Omega_\oplus \approx 28\,\mathrm{AU} \approx 4 \times 10^{12}$ m. Dimension of experiment λ; intrinsic time scale λ/c; then locality postulate fulfilled if $\lambda/\ell \ll 1$. Usually fulfilled (for point coincidences); however, becomes problematic for (extended) waves.
- Decaying muon in storage ring: $a \sim \gamma \frac{v^2}{r} \sim 10^{22} g$; $\ell \sim 10^{-6}$ m:

$$\tau_\mu = \gamma \tau_\mu^0 \left[1 + \frac{2}{3} \left(\frac{\lambda_{\text{Comp}}}{\ell} \right)^2 \right], \qquad \text{corrections} \sim 10^{-16}, \text{ negligible.}$$

- Radiating electron: classical charged particle accelerated by external force $\mathbf{f}$; typical wave length of radiation λ, thus $\lambda \sim \ell$ or $\lambda/\ell \approx 1$, locality violated: Abraham-Lorentz equation

$$m\frac{d\mathbf{v}}{dt} - \frac{2}{3}\frac{e^2}{c^3}\underbrace{\frac{d^2\mathbf{v}}{dt^2}}_{\text{“jerk”}} + \cdots = \mathbf{f}; \qquad \mathbf{x} \text{ and } \mathbf{v} \text{ no longer sufficient.}$$

  Wave phenomena tend to violate the locality postulate (unless we consider the eikonal limit).

- Accelerated system and in-coming electromagnetic wave: Accelerated frame $\mathbf{e}^\alpha = e_i{}^\alpha dx^i$ (with i as coordinate and α as frame index, both $= 0, 1, 2, 3$) obeys

$$\frac{de_i{}^\alpha}{d\tau} = \Phi_\beta{}^\alpha e_i{}^\beta \quad \text{with acc. tensor } \Phi_{\alpha\beta} = (-\mathbf{g}, \mathbf{\Omega}) = -\Phi_{\beta\alpha}.$$

  Plane electromagnetic wave with wave vector $k^i = (\omega, \mathbf{k})$. Observer rotates with Ω_0 around the wave. Then Bahram Mashhoon's analysis yielded $\hat{\omega} = \gamma(\omega \mp \Omega_0) = \omega_{\text{Dop}}(1 \mp \Omega_0/\omega)$. For a wave with spin $\mathbf{s}$, one finds $\sim \mp\gamma \mathbf{s} \cdot \mathbf{\Omega}_0$. This spin-rotation coupling, which has been experimentally verified (GPS), is of a non-local origin. When waves are involved, we are beyond point coincidences and nonlocality sets in.

2. Nonlocal theory of special relativity

- Acceleration induces nonlocality into SR.
- Mashhoon (1993) proposed a nonlocal theory of accelerated systems of the type (memory effect)

$$\mathfrak{F}_{\text{accelerated}}(t) = F_{\text{inertial}}(t) + \int_0^t K(t,t')F_{\text{inertial}}(t')dt'$$

- In 4d, for electrodyn., we have for the excitation $H = (\mathcal{D}, \mathcal{H})$ and the field strength $F = (E, B)$ in comp. (fwh + Yuri Obukhov, Foundations of Classical Electrodynamics, Boston, 2003):

$$H_{\alpha\beta}(\tau,\xi) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} \int d\tau' \underbrace{K_{\alpha\beta}{}^{\gamma\delta}(\tau,\tau')}_{\text{nonlocal kernel}} F_{\gamma\delta}(\tau',\xi)\,.$$

The nonloc. kernel turned out to be (see, however, Mash. 2007)

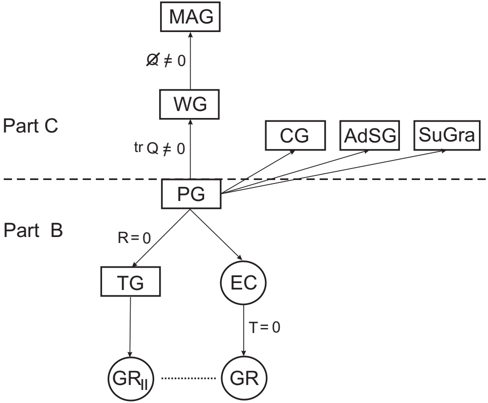
$$K_{\alpha\beta}{}^{\gamma\delta}(\tau,\tau') = \frac{1}{2}\,\epsilon_{\alpha\beta}{}^{\mu[\delta}\left(\delta_\mu^{\gamma]}\,\delta(\tau-\tau') - u\rfloor\Gamma_\mu{}^{\gamma]}(\tau')\right).$$

Connection $\Gamma_\mu{}^\gamma$ with respect to the accelerating frame.

- Conventionally, SR plus *equivalence principle* (EP) $\rightarrow$ GR. Can we apply the EP to the *nonlocal* SR? No, this does not seem to be possible, probably since the EP is a strictly local principle.
- How can we then extend general relativity which is a strictly local theory? Idea: We know electrodynamics is a *gauge theory;* we can make it nonlocal, as shown on the last slide. Take gravity as a gauge theory of translations; generalize it to a nonlocal theory in a similar way as in electrodynamics.
- Poincaré gauge theories of gravity, see arXiv:gr-qc/9602013: Gauging translations $\rightarrow$ Cartan's torsion; gauging Lorentz rotations $\rightarrow$ curvature; Riemann-Cartan spacetime:

GR Curvature	Poincaré gauge theory Curvature + Torsion	$GR_{\|\|}$ Torsion
$\uparrow$	$\longleftarrow$ equivalent $\longrightarrow$	$\uparrow$

- The subcase of the translational gauge theory $GR_{||}$ is equivalent to GR.
- The frontispiece (together with Yuri Obukhov) of Blagojević & FWH, eds., "Gauge Theories of Gravitation," London, 2013, will give an overview of the different gravitational gauge theories:



PG = Poincaré gauge theory, EC = Einstein–Cartan theory, GR = general relativity, TG = translation gauge theory aka teleparallel theory (of gravity), $GR_{||}$ = a specific TG known as teleparallel equivalent of GR (spoken "GR teleparallel"), WG = Weyl(–Cartan) gauge theory, MAG = metric-affine gauge theory, CG = conformal gauge th., AdSG = (anti-)de Sitter gauge theory, SuGra = supergravity (super-Poincaré gauge theory of gravity).
Rectangle $\square \rightarrow$ class of theories; circle $\circ \rightarrow$ definite viable theories; nonmetricity $Q = \not{Q} + \frac{1}{4}(\mathrm{tr}Q)1$, torsion T, curvature R.

3. Analogy between (local) vacuum electrodynamics and ‘Einsteinian’ teleparallelism

Vacuum electrodynamics	$\mathrm{GR}_{\vert\vert}$ (in $\Gamma_\alpha{}^\beta \stackrel{*}{=} 0$): transl. gauge theory
Minkowski spacetime	Weitzenböck spacetime (torsion, flat)
$U(1)$: one generator	T^4: four generators of translations
charge cons. $\Rightarrow$ $dJ = 0$	$d\sigma_\alpha - (e_\alpha \rfloor C^\beta) \wedge \sigma_\beta = 0 \Leftarrow$ energy cons.
$dH = J$	$\boxed{dH_\alpha - E_\alpha = \sigma_\alpha}$ $\Leftarrow$ field eq. of gravity
def. of $F \Rightarrow$ $f_\alpha = (e_\alpha \rfloor F) \wedge J$	$f_\alpha = (e_\alpha \rfloor C^\beta) \wedge \sigma_\beta$ $\Leftarrow$ def. of C^β
$dF = 0$	$dC^\alpha = 0$
F is irreducible $F = dA$	$C^\alpha \stackrel{*}{=} de^\alpha \stackrel{\text{red.}}{\Rightarrow} C^\alpha = \overbrace{{}^{(1)}C^\alpha}^{\text{tensor}} + \overbrace{{}^{(2)}C^\alpha}^{\text{vector}} + \overbrace{{}^{(3)}C^\alpha}^{\text{ax. vec.}}$
$H = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, {}^\star F$	$H_\alpha = \dfrac{1}{2\kappa}\, {}^\star\underbrace{(a_1\, {}^{(1)}C^\alpha + a_2\, {}^{(2)}C^\alpha + a_3\, {}^{(3)}C^\alpha)}_{\mathfrak{C}^\alpha}$
	$a_1 = -1, \quad a_2 = 2, \quad a_3 = \dfrac{1}{2}$
$-\overset{\text{elmg}}{\mathcal{T}}{}_\alpha = e_\alpha \rfloor \overset{\text{elmg}}{V} + (e_\alpha \rfloor F) \wedge H$	$E_\alpha = e_\alpha \rfloor V + (e_\alpha \rfloor C^\beta) \wedge H_\beta$

see M. Blagojević & FWH, *Gauge Theories of Grav.*, London 2013

4. Nonlocal gravitation extends GR

Introduce nonlocality into the framework of teleparallel gravity:

$$\mathcal{H}^{ab}{}_c(x) = \underbrace{\frac{1}{\kappa}\sqrt{-g(x)}\,[\mathfrak{C}^{ab}{}_c(x)}_{\text{this is equivalent to GR}} - \int \underbrace{\Omega^{ai}\Omega^{bj}\Omega_{ck}\,K(x,y)}_{\text{new nonlocal piece}} X_{ij}{}^k(y)\sqrt{-g(y)}\,d^4y]\,,$$

$X_{ij}{}^k := \mathfrak{C}_{ij}{}^k + \frac{1}{3}\check{p}\,\delta^k_{[i}\epsilon_{j]lmn}C^{lmn}$. In 2009, $\check{p} = 0$, in 2014, Mash. chose $\check{p} \neq 0$.

$\Omega(x, y)$ is the world-function of Ruse-Synge (half the square of the geodesic distance connecting x and y); furthermore

$$\Omega_a(x,y) = \frac{\partial\Omega}{\partial x^a}\,,\ \Omega_i(x,y) = \frac{\partial\Omega}{\partial y^i}\,;\ \Omega_{ai} = \frac{\partial\Omega_a}{\partial y^i}\,.\ \text{With } 2\Omega = g^{ab}\Omega_a\Omega_b = g^{ij}\Omega_i\Omega_j\,,$$

and $\Omega_{ai}(x,y) = \Omega_{ia}(x,y)$ with $\lim_{y\to x}\Omega_{ai}(x,y) = -g_{ai}(x)$.

Causal scalar kernel $K(x, y)$ indicates nonlocal deviation from GR. For $K(x, y) = 0$, we recover $\mathrm{GR}_{||}$ and thus, GR. $K(x, y)$ is in general a function of coordinate invariants, as, e.g., of

$$\left({}^{(1)}C_{ij}{}^k\right)^2,\quad \left({}^{(2)}C_{ij}{}^k\right)^2,\quad \left({}^{(3)}C_{ij}{}^k\right)^2.$$

We chose the *simplest* nonlocal constitutive model involving a scalar kernel. The physical origin of this kernel will be discussed below.

5. Linear approximation, dark matter?

Field equation $\quad \partial_b \mathcal{H}^{ab}{}_\mu - \mathcal{E}_\mu{}^a \stackrel{*}{=} t_\mu{}^a$

with grav. energy $\quad \mathcal{E}_\mu{}^a := C_{\mu c}{}^\nu \mathcal{H}^{ac}{}_\nu - \frac{1}{4} e^a{}_\mu (C_{bc}{}^\nu \mathcal{H}^{bc}{}_\nu)$

Lin. approx. $e_i{}^\alpha = \delta_i^\alpha + \psi^\alpha{}_i\,,\ |\psi^\alpha{}_i| \ll 1\,,\ h_{ij} := 2\psi_{(ij)}\,,\ \phi_{ij} := 2\psi_{[ij]}$

Indices become the same. Grav. field strength $C_{ij}{}^k = 2\psi^k{}_{[j,i]}$.

Moreover, $\mathcal{K}(x, y) = K(x - y)$. Then, the constitutive relation reads,

$$\kappa \mathcal{H}^{ij}{}_\alpha(x) = \underbrace{\mathfrak{C}^{ij}{}_\alpha(x)}_{\text{equiv. to GR}} + \int \underbrace{K(x-y)}_{\text{new nonlocal piece}} X^{ij}{}_\alpha(y) d^4 y\,,$$

Scalar kernel $K(x - y)$ is evaluated in Mink. space. With $t_{ij} = {}^\star\sigma_{ij}$,

$$\partial_j \mathfrak{C}^{ij}{}_k + \partial_j \int K(x-y) X^{ij}{}_k(y) d^4 y = \kappa\, t_k{}^i\,.$$

In linear approximation, $\partial_k \mathfrak{C}^{ij}{}_k = {}^0\widetilde{G}^i{}_k$ (= linearized Einstein tensor of the Riemannian space $\hat{=}$ Fierz-Pauli for spin 2).

X_{ijk} is, as we saw, a certain linear combination of the irreducible torsion pieces. In linear approximation,

$$\begin{aligned} X_{ijk} &= \mathfrak{C}_{ijk} + 2\check{p}\,\check{C}_{[i}\eta_{j]k} \qquad \left(\text{with} \quad \check{C}_i := \frac{1}{6}\epsilon_{ijkl}C^{jkl}\right) \\ &= -{}^{(1)}C_{ijk} + 2\,{}^{(2)}C_{ijk} + \frac{1}{2}\left(\eta_{il}\eta_{jm}\eta_{kn} - \frac{2}{3}\check{p}\,\epsilon_{lmn[i}\eta_{j]k}\right){}^{(3)}C^{lmn}\,. \end{aligned}$$

Linearized field equations, cf. Mashhoon, NLG, Eqs.(7.21), (7.22):

$$\left\{ \begin{aligned} {}^{0}\widetilde{G}_{ik} + \partial_j \int K(x-y) X_{(i\cdot k)}{}^{j}(y)\,d^4y &= \kappa\, t_{ik}\,, \\ \partial_j \int K(x-y) X_{[i\cdot k]}{}^{j}(y)\,d^4y &= 0\,. \end{aligned} \right\}$$

$\partial_i t_k{}^i = 0$ follows therefrom. Let us define $\mathfrak{T}_{ik}$ in terms of the nonlocal parts of the field equation:

$$\mathfrak{T}_{ik} := -\frac{1}{\kappa}\partial_j \int K(x-y) X_{(i\cdot k)}{}^{j}(y)\,d^4y\,, \qquad \text{“dark matter”?}$$

Then,

$$\boxed{{}^{0}\widetilde{G}_{ik} = \kappa(t_{ik} + \mathfrak{T}_{ik})} \qquad \text{with} \quad {}^{0}\widetilde{\nabla}_k(t^{ik} + \mathfrak{T}^{ik}) = 0\,.$$

6. Reciprocal kernel

$$
\begin{aligned}
{}^{0}\widetilde{G}_{ik}(x) + \int K(x-y)\, {}^{0}\widetilde{G}_{ik}(y) d^4y &= \kappa t_{ik}(x) + S_{ik}(x) - \check{p} U_{ik}(x) \\
S_{ik}(x) \quad &:= \quad \int \frac{\partial}{\partial y^j} \left[K(x-y)\, \mathfrak{C}_{i\,\cdot\,k}^{\;\,j}(y) \right] d^4y = \dots \\
U_{ik}(x) \quad &:= \quad \partial_j \int K(x-y) \left(\check{C}_i \delta_k^j - \check{C}^j \eta_{ik} \right)(y) d^4y = \dots
\end{aligned}
$$

Fredholm integral eq. of 2nd kind. Solve by the *Liouville-Neumann* method of successive substitutions. Infinite series in terms of iterated kernels $K_n(x-y)$, $n = 1, 2, 3, \dots$:

$$
K_1(x-y) := K(x-y)\,, \quad K_{n+1}(x-y) := -\int K(x-z) K_n(z-y) d^4z\,.
$$

If the resulting infinite series is uniformly convergent, we can define a *reciprocal kernel* $R(x, -y)$ given by $R(x-y) := -\sum_{n=1}^{\infty} K_n(x-y)\,.$ Then the solution of the linearized field equation can be written as

$$
{}^{0}\widetilde{G}_{ik}(x) = \kappa t_{ik}(x) + S_{ik}(x) - \check{p} U_{ik}(x) + \int R(x-y) \left[\kappa t_{ik}(y) + S_{ik}(y) - \check{p} U_{ik}(y) \right] d^4y\,.
$$

Evaluation of these eqs. since 2010 by *Bahram Mashhoon, Carmen Chicone,* Hans-Joachim Blome, *Sohrab Rahvar,* Donato Bini.

7. Newtonian limit

The linearized field equations can now be evaluated with the reciprocal kernel. We find, in the transverse gauge $\partial_k \overline{h}^{ik} = 0$,

$$\Box \overline{h}_{ik} + 2\mathfrak{S}_{ik} = -2\kappa t_{ik} + 2\check{p}\,\mathfrak{U}_{ik},$$

with
$$\overline{h}_{ik} := h_{ik} - \tfrac{1}{2}\eta_{ik} h_l{}^l$$

$$\mathfrak{S}_{ik}(x) := S_{ik}(x) + \int R(x-y) S_{ik} d^4 y$$

$$\mathfrak{U}_{ik}(x) := U_{ik}(x) + \int R(x-y) U_{ik} d^4 y$$

In Newtonian limit $c \to \infty$ and $R(x-y) = \delta(x^0 - y^0) q(\mathbf{x} - \mathbf{y})$, the kernel q is univ., determined by observ. (details in NLG, Sec.7.4):

$$\boxed{\nabla^2 \Phi = 4\pi G(\rho + \rho_{\rm D})\,,} \qquad \rho_{\rm D}(\mathbf{x}) = \int q(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) d^3 y\,.$$

This result, with a suitable q, yields the rotation curves of spiral galaxies! Thus, nonlocality simulates dark matter. Density of dark matter $\rho_{\rm D}$ is a convolution of the density of ordinary matter ρ.

8. Motion of stars in spiral galaxies

- Spiral galaxies, missing mass: Oort (1932), Zwicky (1933)...
- Structure of a spiral galaxy: bulge, disk, globular clusters. According to Kepler's third law (planets around the Sun):

$$\text{Kepler:} \qquad \frac{4\pi^2}{T^2} = \frac{GM}{r^3}\,, \quad v = \frac{2\pi r}{T} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{r}}$$

- However, Rubin + Ford (1970) found flat rotation curves for stars of spiral galaxies: $v(r) \longrightarrow v_0 =$ constant, instead of $\sim 1/\sqrt{r}$
  Review: Sofue + Rubin, Ann. Rev. Astr. Ap. **39** (2001) 137
- Provided Newton-Einstein theory is OK:

$$M = M(r), \qquad M(r) \sim r \Rightarrow \text{ dark matter}$$

$$\text{From} \quad g = \frac{v_0^2}{r}, \quad \text{find} \quad \rho_{\text{Dark}} = -\frac{1}{4\pi G}\vec{\nabla}\cdot\vec{g} \;\Rightarrow\; \rho_{\text{Dark}} = \frac{v_0^2}{4\pi G}\frac{1}{r^2}\,.$$

$$\text{Hence} \quad M_{\text{Dark}} \underbrace{\sim}_{(\text{large } r)} \int \rho_{\text{Dark}} 4\pi r^2 dr = \frac{v_0^2}{G}\, r\,.$$

- But how about abandoning the Newton-Einstein system?

9. The phenomenol. Tohline-Kuhn system

- Recall, Newton's attraction law is not based on a more fundamental principle. He found it in comparing the orbits of planets with the ansatz $\sim 1/r^n$. With $n = 2$ he found agreement.
- Joel Tohline (1983): $g = \frac{GM}{r^2}\left(1 + \frac{r}{\lambda_{\mathrm{TK}}}\right) = \frac{GM}{r^2} + \frac{v_0^2}{r}$

  In general: $\vec{g} = -\vec{\nabla}\Phi\,,\quad \Phi = \underbrace{-\frac{GM}{r}}_{\text{Newton}} + \underbrace{\frac{GM}{\lambda_{\mathrm{TK}}}\ln\left(\frac{r}{\lambda_{\mathrm{TK}}}\right)}_{\text{“dark matter”}}$

  $\lambda_{\mathrm{TK}} = \frac{GM}{v_0^2} =$ fixed constant ~ 1 to 10 kpc (based on rotation curves of spiral galaxies). He sugg. to change Newton's law.
- Jeffrey Kuhn et al. (1987) suggested a a modified Poisson eq.: $\nabla^2\Phi = 4\pi G\left[\rho + \frac{1}{4\pi\lambda_{\mathrm{TK}}}\int\frac{\rho(\vec{y})d^3y}{|\vec{x}-\vec{y}|^2}\right]$ Is OK for spiral galaxies + cluster of galaxies.

  Cf. with Newtonian approx. of NLG: $\boxed{q(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi\lambda_{\mathrm{TK}}}\frac{1}{|\mathbf{x}-\mathbf{y}|^2}}$
- For an actual calculation of q from NLG, see Mashhoon, NLG, Sec.7.4.3. Besides λ_{TK}, two more parameters are required.

10. Discussion

• Rahvar & Mashhoon took 12 nearby spiral galaxies and compared the force law, which was approximately calculated from the Newtonian limit of linearized NLG, with observations. They could determine the three parameters, in particular the Tohline-Kuhn type parameter to 3 ± 2 kpc. The recovery of the Tohline-Kuhn scheme is a nontrivial feature of NLG. These results are very encouraging.

• NLG was developed since Mashhoon wanted to find a nonlocal gravity theory. It came as a total surprise to us that dark matter can be described thereby. Note, Newton's attraction law becomes generalized but Newton's equation of motion (in the non-relativistic approximation), which is much more fundamental (due to its relation to momentum conservation), is left intact—in contrast to MOND.

• Even linearized NLG is not yet exactly solved. No exact solution of NLG other than Minkowski space is known. The decision whether NLG is viable is still pending.

• Can the nonlocal kernel $q(x, y)$ be derived from first principles? Mashhoon observed that the kernel fulfills $\nabla^2 q = 8\pi\lambda q^2$; this corresponds to the *semilinear wave eq.* (Derrick 1964): $\Box\varphi = \varphi^{\mathbf{2}}$.

• The present version of NLG is only globally ('rigidly') Lorentz invariant. Is it possible to determine the kernel of NLG "by insisting that the theory b[e] *locally* Lorentz invariant"?.

• In 2010, we made an ansatz for a nonlocal Poincaré gauge theory, see Blome et al. PRD 2010, Appendix C. NLG would then be a limiting case of a nonlocal Poincaré gauge theory for vanishing curvature. But first, the evaluation of NLG should have absolute priority.

Dear Manuel,
please find below the title an the Abstract of my talk in Tartu.
All best wishes, Friedrich
Nonlocal gravity simulates dark matter
by Friedrich W. Hehl, Univ. of Cologne and Univ. of Missouri-Columbia
Abstract
The analogy between electrodynamics and the translational gauge theory of gravity is employed to develop an ansatz for a nonlocal generalization of Einsteins theory of gravitation. Working in the linear approximation, we show that the resulting nonlocal theory is equivalent to general relativity with dark matter. The nature of the predicted dark matter, which is the manifestation of the nonlocal character of gravity in our model, is briefly discussed. It is demonstrated that this approach can provide a basis for the Tohline-Kuhn treatment of the astrophysical evidence for dark matter.
Literature: F.W. Hehl and B. Mashhoon, Phys. Rev. D 79, 064028 (2009), B. Mashhoon, Nonlocal Gravity, Oxford University Press (2017).

Supplementary material:

Meyer's lemma for teleparallelism (1982)

In PG, we have

$$D\Sigma_\alpha = (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor R^{\beta\gamma}) \wedge \tau_{\beta\gamma}\,,$$
$$D\tau_{\alpha\beta} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} = 0\,.$$

In the case of teleparallelism $R^{\beta\gamma} = 0$.

For *vanishing spin,* $\tau_{\alpha\beta} = 0$, There follows $\vartheta_{[\alpha} \wedge \Sigma_{\beta]} = 0$, that is, the energy-momentum is symmetric; let's call it σ_α. Then,

$$\begin{aligned} D\sigma_\alpha &= (e_\alpha \rfloor T^\beta) \wedge \sigma_\beta \quad \text{or} \\ d\sigma_\alpha &- (\Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta) \wedge \sigma_b = 0 \\ \widetilde{D}\sigma_\alpha &- (-K_\alpha{}^\beta + e_\alpha \rfloor T^\beta) \wedge \sigma_\beta = 0\,. \end{aligned}$$

The contortion, is related to the torsion via $T^\beta = K^\beta{}_\gamma \wedge \vartheta^\gamma$. Substitute:

$$\widetilde{D}\sigma_\alpha - \left[-K_\alpha{}^\beta + e_\alpha \rfloor (K^\beta{}_\gamma \wedge \vartheta^\gamma)\right] \wedge \sigma_\beta = 0 \quad \text{or}$$
$$\widetilde{D}\sigma_\alpha + \left[K_\alpha{}^\beta - (e_\alpha \rfloor K^\beta{}_\gamma)\vartheta^\gamma + K^\beta{}_\gamma \delta^\gamma_\alpha\right] \wedge \sigma_\beta = 0\,.$$

Because of $K_{(\alpha\beta)} = 0$, we find

$$\widetilde{D}\sigma_\alpha - (e_\alpha \rfloor K^{\beta\gamma})\vartheta_{[\gamma} \wedge \sigma_{b]} = \boxed{\widetilde{D}\sigma_\alpha = 0}\,.$$