Mini-Superspace Quantum Supergravity and its Hidden Hyperbolic Kac-Moody Structures

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work with Th. Damour (IHES)
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arXiv : 1406.1309, *Quantum Supersymmetric Bianchi IX Cosmology*]
569. Wilhelm and Else Heraeus Seminar
Quantum **C**osmology

Short version (less than a minute!)

We study the quantum dynamics of a supersymmetric "squashed three-sphere" by dimensionally reducing (to one timelike dimension) the action of $D = 4$ simple supergravity for an $SO(3)$-homogeneous (Bianchi IX) cosmological model.

The quantization of the homogeneous gravitino field leads to a 64-dimensional fermionic Hilbert space.

The algebra of the supersymmetry and the Hamiltonian constraints is found to close.

The quantum Hamiltonian is built from operators that generate a 64-dimensional representation of the (infinite-dimensional) maximally compact sub-algebra of the rank-3 hyperbolic Kac-Moody algebra AE_3. Some exponentials of these operators generate a spinorial extension of the Weyl group of AE_3 which describe (in the small wavelength limit) the chaotic quantum evolution of the Universe near the cosmological singularity.

The fine structure of the dynamic on the Hilbert space is obtained : solutions split between discrete and continuous modes (depending on arbitrary constants or functions)

Part of the wave function may avoid the cosmological singularity.

PLAN

E_{10} (OR AE_n) SEEMS TO BE HIDDEN IN (SUPER)GRAVITY

T. Damour and M. Henneaux, Phys. Rev. Lett. **86**, 4749 (2001),
T. Damour, M. Henneaux, B. Julia and H. Nicolai, Phys. Lett. B **509**, 323 (2001),
T. Damour, M. Henneaux and H. Nicolai, Phys. Rev. Lett. **89**, 221601 (2002),
T. Damour, A. Kleinschmidt and H. Nicolai, Phys. Lett. B **634**, 319 (2006),
S. de Buyl, M. Henneaux and L. Paulot, JHEP **0602**, 056 (2006), T. Damour, A. Kleinschmidt and H. Nicolai, JHEP **0608**, 046 (2006), ...

[first conjectured by Julia '82; related conjectures Ganor '99, '04; West (E_{11}) 01]

Lead to Gravity/Coset conjecture (DHN, 2002)

'Duality' between $D = 11$ supergravity (or, hopefully, M-theory) and the (quantum) dynamics of a massless spinning particle on $E_{10}/K(E_{10})$

KAC-MOODY ALGEBRAS (IN VERY, VERY BRIEF)

Generalized Cartan matrix :

$$(A_{i\,i}) = 2\,,\ A_{i\,j} \in \mathbb{Z}^- \ \textit{if}\ i \neq j \quad , \quad A_{i\,j} = 0\ \textit{iff}\ A_{j\,i} = 0\,,\ A = D\,S$$

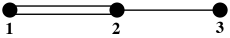
with D diagonal positive and S symmetric.

Chevaley–Serre presentation :

$$[h_i, h_j] = 0\ \ [h_i, e_j] = A_{i\,j}\,e_j\ (\textit{no sum.})\,, [h_i, f_j] = -A_{i\,j} f_j\ ,\ [e_i, f_j] = \delta_{i\,j}\,h_j\ ,\ \mathrm{ad}_{e_i}^{1-A_{i\,j}}(e_j) = 0$$

S positive corresponds to finite dimensionnal semi-simple Lie algebras.

AE_3: $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$



Level decomposition: $\alpha = \ell\,\alpha_1 + m_2\,\alpha_2 + m_3\,\alpha_3$

Kac-Moody algebras (in very, very brief)

Chevalley involution :

$$\omega(e_i) = -f_i \quad , \quad \omega(f_i) = -e_i \quad , \quad \omega(h_i) = -h_i$$

"Maximally compact" sub-algebra :

$$\omega(x) = x \quad , \quad \textit{generated by} \quad x_i = e_i - f_i$$

Coset element

$$\mathcal{V} = \exp[\sum_{a=1}^{10} \beta^a H_{\hat{a}}] \exp[\sum_{\alpha\in\Delta^+} \sum_{s=1}^{mult(\alpha)} \nu_{\alpha,s} E_\alpha^{(s)} \in E_{10}/K(E_{10})$$

$$\dot{\mathcal{V}}\,\mathcal{V}^{-1} =: \mathcal{P} + \mathcal{Q} \quad , \quad \mathcal{Q} \in K(E_{10}) \quad , \quad \mathcal{P} \in E_{10} \ominus K(E_{10})$$

$h = \beta^i H_i \quad , \quad \alpha = n_k \alpha^k \quad , \quad [h, E_\alpha^{(s)}] = \alpha(h)\, E_\alpha^{(s)} \quad , \quad \alpha(h) = n_k\, \alpha^k(h) = n_k\, A_{ji}\, \beta^j$

$\mathcal{P} := (\dot{\mathcal{V}}\,\mathcal{V}^{-1})^{sym}$: coset velocity

$\mathcal{Q} := (\dot{\mathcal{V}}\,\mathcal{V}^{-1})^{antisym}$: "K"–angular velocity

GRAVITY/COSET CORRESPONDENCE

E_{10} : Damour, Henneaux, Nicolai '02; related: Ganor '99 '04; E_{11} : West '01, de Buyl, Hennaux, Paulot '06, Damour, Hillmann '09

$$G_{\mu\nu}(t,\mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t,\mathbf{x}), \psi_\mu(t,\mathbf{x})$$

$$S_{11} = \int d_x^{11}\{ \tfrac{E}{4} R(G) - \tfrac{E}{48} (d\mathcal{A}_3)^2 + \ldots \}$$

Gradient Expansion (BKL)
($\sim$ Small Tension Expansion: $\alpha' \to \infty$)

$$\partial_{x^1}^{k_1} \partial_{x^2}^{k_2} \ldots \partial_{x^{10}}^{k_{10}} \ll \partial_T^{k_1+k_2+\ldots+k_{10}}$$

$$S_1^{\text{COSET}} = \int dt \{ \tfrac{1}{4\,n(t)} \langle P(t), P(t) \rangle - \tfrac{i}{2} (\Psi(t) \mid \mathcal{D}^{\text{vs}}\, \Psi(t))_{\text{vs}} + \ldots \}$$

Height Expansion
in Kac-Moody Algebra

$$\langle P(t), P(t) \rangle = 0$$

$$\partial_t \mathcal{P}(t) = [\mathcal{Q}(t),\, \mathcal{P}(t)] \quad , \quad \partial_t \Psi(t) = \mathcal{Q}^{vs}\, \Psi(t)$$

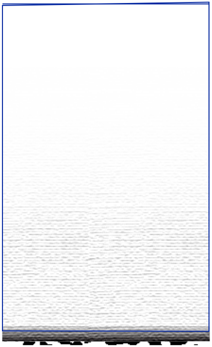
$$\mathcal{Q}(t)|_\alpha \propto \cosh^{-1}[2\sqrt{E}(t-t_c)] J_\alpha$$

$$E = -\frac{1}{2} G_{ab} \dot{\beta}_{\|}^a \dot{\beta}_{\|}^a \quad , \quad J_\alpha = E_\alpha - E_{-\alpha}$$

V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).

Gravity/Coset Correspondence

Near a spacelike singularity



Singularity

$\mathcal{R} << \ell_P^2$

SUGRA regime

$G_{\mu\nu}(x^\alpha),\ \Psi_\mu(x^\alpha)$

$\mathcal{R} >> \ell_P^2$

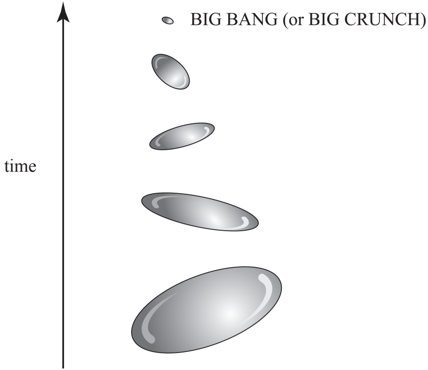
COSET regime

$E_{10}/K[E_{10}]$

A CONCRETE CASE STUDY

T. Damour, Ph. S. , arXiv : 1304.6381, Class. Quantum Grav. 30 (2013) 162001,
1406.1309, *Quantum Supersymmetric Bianchi IX Cosmology*

• Quantum dynamics of a supersymmetric triaxially squashed three-sphere



Evolution of the eigendirections of the curvature tensor.

Notice that the space remains homogeneous, it just loses its isotropy

- Rarita-Schwinger Lagrangian

$$\mathcal{L}_{RS} = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\overline{\psi}_\alpha\gamma_5\gamma_\beta\mathcal{D}_\gamma\psi_\delta$$

- Einstein-Hilbert Lagrangian

$$\mathcal{L}_{EH} = \frac{1}{2\kappa^2}\sqrt{{}^4g}\,R = -\frac{1}{8\kappa^2}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta}\theta^{\hat\gamma}_\rho\theta^{\hat\delta}_\sigma R^{\hat\alpha\hat\beta}{}_{\mu\nu}(\omega)$$

- Connexion with torsion

$$\omega_{\hat\alpha\hat\beta\mu} = \overset{0}{\omega}_{\hat\alpha\hat\beta\mu} + \kappa_{\hat\alpha\hat\beta\mu}$$

$$\kappa_{\hat\alpha\hat\beta\hat\gamma} = \kappa_{\hat\alpha\hat\beta\mu}\,\theta^\mu_{\hat\gamma} = \frac{\kappa^2}{4}\left(\overline{\psi}_{\hat\beta}\gamma_{\hat\alpha}\psi_{\hat\gamma} - \overline{\psi}_{\hat\alpha}\gamma_{\hat\beta}\psi_{\hat\gamma} + \overline{\psi}_{\hat\beta}\gamma_{\hat\gamma}\psi_{\hat\alpha}\right)$$

- Total Lagrangian

$$\mathcal{L}_{Tot} = \theta\left[\frac{1}{2}\overset{0}{R} + \overset{0}{L}_{3/2} + \frac{1}{8}T^{\hat\alpha}T_{\hat\alpha} - \frac{1}{16}T^{\hat\alpha\hat\beta\hat\gamma}T_{\hat\gamma\hat\beta\hat\alpha} - \frac{1}{32}T^{\hat\alpha\hat\beta\hat\gamma}T_{\hat\alpha\hat\beta\hat\gamma}\right]$$

with

$$T_{\hat\alpha\hat\beta\hat\gamma} := \overline{\psi}_{\hat\beta}\gamma_{\hat\alpha}\psi_{\hat\gamma} \qquad , \qquad T_{\hat\alpha} = \overline{\psi}_{\hat\alpha}\gamma^{\hat\beta}\psi_{\hat\beta}$$

Technically: Reduction to one, time-like, dimension of the action of $D=4$ simple supergravity built on an $SO(3)$-homogeneous (Bianchi IX) cosmological model

$$g_{\mu\nu}\, dx^\mu\, dx^\nu \quad = \quad -N^2(t)dt^2 + g_{ab}(t)(\tau^a(x) + N^a(t)dt)(\tau^b(x) + N^b(t)dt)\,,$$

τ^a: left-invariant one-forms on $SU(2) \approx S_3 : d\tau^a = \frac{1}{2}\, \varepsilon^a{}_{bc}\, \tau^b \wedge \tau^c$

DYNAMICAL DEGREES OF FREEDOM

• Gauss-decomposition of the metric:

$$g_{bc} = \sum_{\widehat{a}=1}^{3} e^{-2\beta^{a}} \, S^{\widehat{a}}{}_{b}(\varphi_1, \varphi_2, \varphi_3) \, S^{\widehat{a}}{}_{c}(\varphi_1, \varphi_2, \varphi_3)$$

Six metric degrees of freedom :
cologarithms of the squashing parameters of the 3-sphere

$$\beta^{a} = (\beta^{1}(t), \beta^{2}(t), \beta^{3}(t))$$

and three Euler angles:

$$\varphi_a = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$$

Fermionic Degrees of Freedom

Gravitino components ($\psi_A^{\widehat{\alpha}}$) in specific gauge-fixed orthonormal frame $\theta^{\widehat{\alpha}}$ canonically associated to the Gauss-decomposition :

$$\theta^{\widehat{0}} = N(t)dt\,,\ \theta^{\widehat{a}} = \sum_b e^{-\beta^a(t)} S^{\widehat{a}}{}_b(\varphi_c(t))(\tau^b(x) + N^b(t)dt) \qquad .$$

After a suitable redefinition of the gravitino field :

- 3×4 dynamical gravitino components $\Phi_A^a := (\gamma^{\hat{a}}\, g^{\frac{1}{4}} \psi^{\hat{a}})_A$, $a = 1, 2, 3; A = 1, 2, 3, 4$

so that $g^{\frac{1}{2}}\, \bar{\psi}_{\widehat{a}} \gamma^{\widehat{a}\widehat{0}\widehat{b}} \dot{\psi}_{\widehat{b}} = G_{ab}\, \Phi^a\, \dot{\Phi}^b$

- and four Lagrange multipliers : $\Psi_A'^{\widehat{0}} := g^{\frac{1}{4}} (\psi^{\widehat{0}} - \sum_a \gamma^{\widehat{0}\widehat{a}}\, \psi_{\widehat{a}})_A$.

SUPERSYMMETRIC ACTION (FIRST ORDER FORM)

$$S = \int dt \left[\pi_a \, \dot{\beta}^a + p_{\theta^a} \, \dot{\theta}^a + \frac{i}{2} G_{ab} \, \Phi^a_A \, \dot{\Phi}^b_A + \bar{\Psi}'^A_{\hat{0}} \, \mathcal{S}_A - \tilde{N} H - N^a H_a \right]$$

G_{ab}: Lorentzian-signature quadratic form:

$$G_{ab} \, d\beta^a \, d\beta^b \equiv \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a \right)^2$$

G_{ab} defines the kinetic terms of the gravitino, as well as those of the β^a's:

$$\frac{1}{2} G_{ab} \, \dot{\beta}^a \, \dot{\beta}^b$$

Lagrange multipliers imply constraints : $\mathcal{S}_A \approx 0$, $H \approx 0$, $H_a \approx 0$

Suppose to have a fermionic Lagrangian : $\mathcal{L}_F = \frac{1}{2}\phi_A M^{AB}\dot{\phi}_B$ *with* $M^{AB} = +M^{BA}$
We obtain constraints χ^A linking the ϕ_A variables and their conjugate momenta ϖ^A:

$$\varpi^B := \frac{\partial^L \mathcal{L}_F}{\partial \dot{\phi}_B} = -\frac{1}{2}\phi_A M^{AB} \quad , \quad \{\varpi^B, \phi_A\} = -\delta^B_A \quad , \quad \chi^B := \varpi^B + \frac{1}{2}\phi_A M^{AB}$$

$$\{\chi^A, \chi^B\} = \{\varpi^A, \frac{1}{2}\phi_K M^{KB}\} + \{\frac{1}{2}\phi_K M^{KA}, \varpi^B\} = -\frac{1}{2}(M^{AB} + M^{BA}) = -M^{AB}$$

Assuming M^{AB} invertible, the Dirac bracket reads

$$\{\ ,\ \}_D = \{\ ,\ \} - \{\ ,\chi^A\}(-)M_{AB}\{\chi^B,\ \}$$

and

$$\{\phi_A, \phi_B\}_D = \{\phi_A, \chi^K\}M_{KL}\{\chi^L, \phi_B\} = +M_{AB} \quad , \quad \{\hat{\phi}_A, \hat{\phi}_B\} = +i\, M_{AB}$$

It is the matrix M_{AB} (not necessarily positive defined) that defines the scalar product in the Hilbert space on which the operators representing the quantum fields act.
Conjugaison has to be reconsidered :

$$\langle u|Op|v\rangle_h = \overline{u}^T h\, Op\, v = \langle u|Op^{\ddagger}|v\rangle_h = \overline{Op\, u}^T\, h\, v = \overline{u}^T \overline{Op}^T\, h\, v$$

$$\Rightarrow \quad Op^{\ddagger} = h^{-1} Op^{\dagger} h$$

QUANTIZATION

• Bosonic dof:

$$\widehat{\pi}_a = -i\,\frac{\partial}{\partial\beta^a}\;;\quad \widehat{p}_{\varphi^a} = -i\,\frac{\partial}{\partial\varphi^a}$$

• Fermionic dof:

$$\widehat{\Phi}^a_A\,\widehat{\Phi}^b_B + \widehat{\Phi}^b_B\,\widehat{\Phi}^a_A = G^{ab}\,\delta_{AB}$$

This is the Clifford algebra Spin $(8^+, 4^-)$

• The wave function of the Universe $\Psi_\sigma(\beta^a, \theta^a)$ is a 64-dimensional spinor of Spin $(8, 4)$ and the gravitino operators Φ^a_A are 64×64 "gamma matrices" acting on Ψ_σ, $\sigma = 1, \ldots, 64$. Hermiticity $(\ddagger)$ is defined by use of $h = \Gamma_9\Gamma_{10}\Gamma_{11}\Gamma_{12} = h^{-1}$: $\Psi^\ddagger := h\,\Psi^\dagger\,h$.

Dirac Quantization of the Constraints

$$\Psi = \Psi[\beta, \varphi]\,, \quad \widehat{\mathcal{S}}_A\,\Psi = 0\,, \quad \widehat{H}\,\Psi = 0\,, \quad \widehat{H}_a\,\Psi = 0$$

Diffeomorphism constraints $\Leftrightarrow \widehat{p}_{\varphi^a}\,\Psi = -i\,\dfrac{\partial}{\partial\varphi^a}\,\Psi = 0$

S-wave function $\Psi(\beta^a)$ submitted to constraints

$$\widehat{\mathcal{S}}_A(\widehat{\pi}, \beta, \widehat{\Phi})\,\Psi(\beta) = 0\,, \quad \widehat{H}(\widehat{\pi}, \beta, \widehat{\Phi})\,\Psi(\beta) = 0$$

$\widehat{\pi}_a = -i\,\frac{\partial}{\partial\beta^a} \Rightarrow 4 \times 64 + 64$ PDE's for the 64 functions $\Psi_\sigma(\beta^1, \beta^2, \beta^3)$

Heavily overdetermined system of PDE's

Explicit form of the SUSY constraints

$$\begin{aligned}\widehat{\mathcal{S}}_A &= -\frac{1}{2}\sum_a \widehat{\pi}_a\, \Phi^a_A + \frac{1}{2}\sum_a e^{-2\beta^a}(\gamma^5\,\Phi^a)_A \\ &- \frac{1}{8}\coth\beta_{12}(\widehat{S}_{12}(\gamma^{12}\,\widehat{\Phi}^{12})_A + (\gamma^{12}\,\widehat{\Phi}^{12})_A\,\widehat{S}_{12}) \\ &+ \text{cyclic}_{(123)} + \frac{1}{2}(\widehat{\mathcal{S}}^{\text{cubic}}_A + \widehat{\mathcal{S}}^{\text{cubic}\,\dagger}_A)\end{aligned}$$

where $\gamma^5 \equiv \gamma^{\hat{0}\hat{1}\hat{2}\hat{3}}$, $\beta_{12} \equiv \beta^1 - \beta^2$, $\widehat{\Phi}^{12} \equiv \widehat{\Phi}^1 - \widehat{\Phi}^2$,

$$\begin{aligned}\widehat{S}_{12}(\widehat{\Phi}) &= \frac{1}{2}[(\widehat{\overline{\Phi}}^3\,\gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1 + \widehat{\Phi}^2)) + (\widehat{\overline{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^1) \\ &+ (\widehat{\overline{\Phi}}^2\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2) - (\widehat{\overline{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)]\,,\end{aligned}$$

$$\begin{aligned}\widehat{\mathcal{S}}_A^{\rm cubic} &= \frac{1}{4}\sum_a (\widehat{\overline{\Psi}}_{0'}\,\gamma^{\widehat{0}}\,\widehat{\Psi}_{\widehat{a}})\,\gamma^{\widehat{0}}\,\widehat{\Psi}^A_{\widehat{a}} - \frac{1}{8}\sum_{a,b} (\widehat{\overline{\Psi}}_{\widehat{a}}\,\gamma^{\widehat{0}}\,\widehat{\Psi}_{\widehat{b}})\,\gamma^{\widehat{a}}\,\widehat{\Psi}^A_{\widehat{b}} \\ &+ \frac{1}{8}\sum_{a,b} (\widehat{\overline{\Psi}}_{0'}\,\gamma^{\widehat{a}}\,\Psi_{\widehat{b}})(\gamma^{\widehat{a}}\,\Psi^A_{\widehat{b}} + \gamma^{\widehat{b}}\,\Psi^A_{\widehat{a}})\,,\end{aligned}$$

where $\Psi_{\widehat{0}'} = \gamma_{\widehat{0}}(\Phi^1 + \Phi^2 + \Phi^3)$,

(Open) Superalgebra satisfied by the $\widehat{\mathcal{S}}_A$'s and $\widehat{H}$

This unique, hermitian ordering of $\mathcal{S}_A$ defines a unique Hamiltonian, such that :

$$\widehat{\mathcal{S}}_A\,\widehat{\mathcal{S}}_B + \widehat{\mathcal{S}}_B\,\widehat{\mathcal{S}}_A = 4\,i\sum_C \widehat{L}_{AB}^{C}(\beta)\,\widehat{\mathcal{S}}_C + \frac{1}{2}\,\widehat{H}\delta_{AB}$$

$$[\widehat{\mathcal{S}}_A, \widehat{H}] = \widehat{M}_A^B\,\widehat{\mathcal{S}}_B + \widehat{N}_A\widehat{H}$$

Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\,\widehat{H} = G^{ab}(\widehat{\pi}_a + i\,A_a)(\widehat{\pi}_b + i\,A_b) + \widehat{\mu}^2 + \widehat{W}(\beta)\,,$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

$$A_a = \partial_a \ln[e^{\frac{3}{4}\beta^0}(\sinh\beta_{12}\ \sinh\beta_{23}\ \sinh\beta_{31})^{-\frac{1}{8}}] \qquad \textit{pure gauge vector potential}$$

$$\widehat{W}(\beta) = W_g^{\text{bos}}(\beta) + \widehat{W}_g^{\text{spin}}(\beta) + \widehat{W}_{\text{sym}}^{\text{spin}}(\beta)\,.$$

$$W_g^{\text{bos}}(\beta) = \frac{1}{2}\,e^{-4\beta^1} - e^{-2(\beta^2+\beta^3)} + \text{cyclic}_{123}$$

Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$\begin{aligned}\widehat{W}_g^{\text{spin}}(\beta,\widehat{\Phi}) &= e^{-\alpha_{11}^g(\beta)}\widehat{J}_{11}(\widehat{\Phi}) + e^{-\alpha_{22}^g(\beta)}\widehat{J}_{22}(\widehat{\Phi}) \\ &+ e^{-\alpha_{33}^g(\beta)}\widehat{J}_{33}(\widehat{\Phi})\,.\end{aligned}$$

Linear forms $\alpha_{ab}^g(\beta) = \beta^a + \beta^b \Leftrightarrow$ six level-1 roots of AE_3

$$\widehat{W}_{\text{sym}}^{\text{spin}}(\beta) = \frac{1}{2}\,\frac{(\widehat{S}_{12}(\widehat{\Phi}))^2 - 1}{\sinh^2 \alpha_{12}^{\text{sym}}(\beta)} + \text{cyclic}_{123}\,,$$

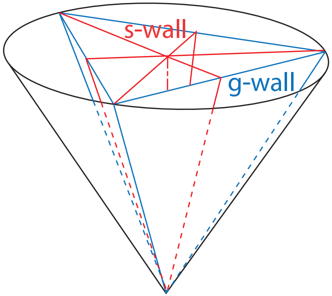
Linear forms $\alpha_{12}^{\text{sym}}(\beta) = \beta^1 - \beta^2$, $\alpha_{23}^{\text{sym}}(\beta) = \beta^2 - \beta^3$, $\alpha_{31}^{\text{sym}}(\beta) = \beta^3 - \beta^1 \Leftrightarrow$ three level-0 roots of AE_3

SPIN DEPENDENT (CLIFFORD) OPERATORS COUPLED TO AE_3 ROOTS

$$\widehat{S}_{12}(\widehat{\Phi}) = \frac{1}{2}[(\overline{\widehat{\Phi}}^3\,\gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1 + \widehat{\Phi}^2)) + (\overline{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^1) + (\overline{\widehat{\Phi}}^2\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2) - (\overline{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)]\,,$$

$$\widehat{J}_{11}(\widehat{\Phi}) = \frac{1}{2}\,[\overline{\widehat{\Phi}}^1\gamma^{\hat{1}\hat{2}\hat{3}}(4\widehat{\Phi}^1 + \widehat{\Phi}^2 + \widehat{\Phi}^3) + \overline{\widehat{\Phi}}^2\,\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^3]\,.$$

• $\widehat{S}_{12}$, $\widehat{S}_{23}$, $\widehat{S}_{31}, \widehat{J}_{11}, \widehat{J}_{22}, \widehat{J}_{33}$ generate (via commutators) a 64-dimensional representation of the (infinite-dimensional) "maximally compact" sub-algebra $K(AE_3) \subset AE_3$. [The fixed set of the (linear) Chevalley involution, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h_i) = -h_i$, which is generated by $x_i = e_i - f_i$.]



Lorentzian structure of the β-space.
$\beta^0 := \beta^1 + \beta^2 + \beta^3$ play the rôle of time

Symmetry walls $\sinh^{-2}(\beta^i - \beta^j)$:
$\beta^1 - \beta^2 = 0,\ \beta^2 - \beta^3 = 0,\ \beta^3 - \beta^1 = 0$

Gravitational walls $\exp(-2\,\beta^i)$:
$\beta^1 = 0,\ \beta^2 = 0,\ \beta^3 = 0$

$$\begin{array}{l}\{\Phi^k_A\,\Phi^l_B+\Phi^l_B\,\Phi^k_A\}=G^{kl}\delta_{AB}Id_{64}\\ b^k_+=\Phi^k_1+i\,\Phi^k_2\qquad,\qquad b^k_-=\Phi^k_3-i\,\Phi^k_4\\ \tilde b^k_+=\Phi^k_1-i\,\Phi^k_2\qquad,\qquad \tilde b^k_-=\Phi^k_3+i\,\Phi^k_4\\ \{b^k_\epsilon\,,\,\tilde b^l_\sigma\}=2\,G^{kl}\delta_{\epsilon\sigma}\,Id_{64}\end{array}$$

$$\mathcal{S}_\epsilon=\frac{i}{2}\partial_{\beta_k}b^k_\epsilon+\alpha_k\,b^k_\epsilon+\frac{1}{2}\mu_{klm}\,B^{klm}_\epsilon+\rho_{klm}C^{klm}_\epsilon+\frac{1}{2}\nu_{klm}\,D^{klm}_\epsilon$$

where

$$\begin{array}{lcl}B^{klm}_\epsilon&=&b^k_\epsilon\,b^l_\epsilon\,\tilde b^m_\epsilon-G^{lm}\,b^k_\epsilon+G^{km}b^l_\epsilon\\ C^{klm}_\epsilon&=&b^k_\epsilon\,b^l_{-\epsilon}\,\tilde b^m_{-\epsilon}-G^{lm}\,b^k_\epsilon\\ D^{klm}_\epsilon&=&b^k_{-\epsilon}\,b^l_{-\epsilon}\,\tilde b^m_\epsilon\end{array}$$

and all the tensor components are purely imaginary (as they must be to insure hermiticity of the operators).

Fermion number operator : $\widehat{N}_F:=G_{ab}\tilde b^a_+b^b_+ + G_{ab}\tilde b^a_-b^b_- = \frac{1}{2}G_{ab}\overline{\widehat{\Phi}}^a\gamma^{\widehat{1}\widehat{2}\widehat{3}}\widehat{\Phi}^b+3=:\widehat{C}_F+3$

SOLUTIONS OF SUSY CONSTRAINTS

Overdetermined system of 4×64 Dirac-like equations

$$\widehat{\mathcal{S}}_A \Psi = \left(\frac{i}{2} \Phi_A^a \frac{\partial}{\partial \beta^a} + \ldots \right) \Psi = 0$$

Space of solutions splits according to the fermion number $N_F = C_F + 3)$

Depending on the fermion number there exist "discrete states" and "continuous states" (parametrized by initial data involving arbitrary *functions*, at $C_F = -1, 0, +1)$

BI-COMPLEX

The wave function we are looking for are solutions of :

$$\mathcal{S}_\epsilon\,\Psi = 0 = \tilde{\mathcal{S}}_\epsilon\,\Psi$$

There is a unique lower state :

$$b^k_\epsilon \Psi_0 = 0 \quad , \quad N_F\,\Psi_0 = 0 \quad , \quad \mathcal{S}_\epsilon\; N_F = (N_F - 1)\,\mathcal{S}_\epsilon \quad , \quad \tilde{\mathcal{S}}_\epsilon\; N_F = (N_F + 1)\,\tilde{\mathcal{S}}_\epsilon$$

- Level 0 : Ψ_0 (1)
- Level 1 : $\{\tilde{b}^k_+,\ \tilde{b}^l_-\}\Psi_0$ (2×3)
- Level 2 : $\{\tilde{b}^{(k}_+\tilde{b}^{l)}_-,\ \tilde{b}^{[k}_+\tilde{b}^{l]}_+,\ \tilde{b}^{[k}_+\tilde{b}^{l]}_-,\ \tilde{b}^{[k}_-\tilde{b}^{l]}_-\}\Psi_0$ (6+3+3+3)
- Level 3 : $\{\frac{1}{2}\epsilon^{[a}{}_{k\,l}\tilde{b}^{m]}_-\tilde{b}^k_+\tilde{b}^l_+,\ \tilde{b}^1_-\tilde{b}^2_-\tilde{b}^3_-,\ \frac{1}{2}\epsilon^{(a}{}_{k\,l}\tilde{b}^{m)}_-\tilde{b}^k_+\tilde{b}^l_+\}\Psi_0,\ 2 \times (3+1+6)$
- Level 4 : . . . (15)
- Level 5 : . . . (6)
- Level 6 : $\tilde{b}^1_+\tilde{b}^2_+\tilde{b}^3_+\tilde{b}^1_-\tilde{b}^2_-\tilde{b}^3_-\Psi_0$ (1)

EXPLICIT EQUATIONS AND SOLUTIONS

With $x := e^{2\beta_1},\ y := e^{2\beta_2},\ z := e^{2\beta_3},$:

- Level 0 :

$$\frac{i}{2}\partial_{\beta_k} f - \phi_k f = 0$$

$$\phi_k = -i\left\{\frac{1}{2} - \frac{1}{2\,x} - \frac{3}{8}\frac{y(x-z)+z(x-y)}{(x-y)(x-z)}, \textit{cyclic perm.}\right\}$$

$$f = f_0\, Exp[-\frac{1}{2}(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})](x\,y\,z)^{-\frac{5}{4}}\,((x-y)(y-z)(z-x))^{3/8}$$

Differs from previously obtained solutions by the factors vanishing on the symmetry walls

Explicit Equations and Solutions

- Level 1 : $\Psi = f_k^{\sigma}\, \tilde{b}_{\sigma}^{k}\, \Psi_0$

$$\mathcal{S}_\epsilon \Psi = 0 \Leftrightarrow \frac{i}{2}\partial_{\beta_k} f_\epsilon^k + \varphi_k f_\epsilon^k = 0$$

$$\tilde{\mathcal{S}}_\epsilon \Psi = 0 \Leftrightarrow \begin{cases} \tilde{\nu}_{[kl]}{}^{m} f_m^{\epsilon} = 0 \\ \left(\frac{i}{2}\partial_{\beta_{[k}} f_{l]}^{\epsilon} + \tilde{\varphi}_{[k} f_{l]}^{\epsilon} - 2\,\tilde{\mu}_{[kl]}{}^{m} f_m^{\epsilon} = 0\right) \\ \frac{i}{2}\partial_{\beta_k} f_l^{\epsilon} + \tilde{\varphi}_k f_l^{\epsilon} - 2\,\tilde{\rho}_{k\,l}{}^{m} f_m^{\epsilon} = 0 \end{cases}$$

$$f_k^{\epsilon} = f^{\epsilon}\,\{x(y-z),\, y(z-x),\, z(x-y)\}$$

with

$$f^{\epsilon} = f_0^{\epsilon}\, Exp[-\frac{1}{2}(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})](x\,y\,z)^{-\frac{3}{4}}\,((x-y)(y-z)(z-x))^{-3/8}$$

Two-dimensionnal space of solutions

Previous works concluded to the absence of states with odd fermion numbers

Explicit Equations and Solutions

- Level 2 : $\Psi = \frac{1}{2} f_{p\,q}^{\epsilon,\epsilon'}\, \tilde{b}^p_\epsilon \tilde{b}^q_{\epsilon'}\, \Psi_0$, with $f_{p\,q}^{\epsilon,\,\epsilon'} = -f_{q\,p}^{\epsilon',\,\epsilon}$)

$$\mathcal{S}_\epsilon \Psi = 0 \Leftrightarrow \begin{cases} \frac{i}{2}\partial_{\beta_k} G^{kp} f_{p\,n}^{\epsilon,\epsilon} + \varphi^k f_{k\,n}^{\epsilon,\epsilon} - \mu^{k\,l}{}_n f_{k\,l}^{\epsilon,\epsilon} - \nu^{k\,l}{}_n f_{k\,l}^{-\epsilon,-\epsilon} = 0 \\ \frac{i}{2}\partial_{\beta_k} G^{kp} f_{p\,n}^{\epsilon,-\epsilon} + \varphi^k f_{k\,n}^{\epsilon,-\epsilon} - 2\,\rho^{k\,l}{}_n f_{k\,l}^{\epsilon,-\epsilon} = 0 \end{cases}$$

$$\tilde{\mathcal{S}}_\epsilon \Psi = 0 \Leftrightarrow \begin{cases} \left[\frac{i}{2}\partial_{\beta_n} f_{p\,q}^{\epsilon,\epsilon} + \tilde{\varphi}_n f_{p\,q}^{\epsilon,\epsilon} + 2\,\tilde{\mu}_{p\,q}{}^s f_{n\,s}^{\epsilon,\epsilon}\right] \varepsilon^{n\,p\,q} = 0 \\ \frac{i}{2}\partial_{\beta_{[k}} f_{l]\,q}^{\epsilon,-\epsilon} + \tilde{\varphi}_{[k} f_{l]q}^{\epsilon,-\epsilon} - \tilde{\mu}_{k\,l}{}^s f_{s\,q}^{\epsilon,-\epsilon} - 2\,\tilde{\rho}_{[k\,|q|}{}^s f_{l]\,s}^{\epsilon,-\epsilon} = 0 \\ \frac{i}{2}\partial_{\beta_k} f_{p\,q}^{-\epsilon,-\epsilon} + \tilde{\varphi}_k f_{p\,q}^{-\epsilon,-\epsilon} + 4\,\tilde{\rho}_{k\,[p}{}^s f_{q]\,s}^{-\epsilon,-\epsilon} + 2\,\tilde{\nu}_{p\,q}{}^s f_{k\,s}^{\epsilon,\epsilon} = 0 \\ \tilde{\nu}_{[p\,q}{}^s f_{n]\,s}^{-\epsilon,\epsilon} = 0 \end{cases}$$

Equations for the $f_{p\,q}^{\epsilon\,\epsilon}$ and $f_{p\,q}^{\epsilon\,-\epsilon}$ components decouple.

- Level 2 : three discrete modes ($\epsilon = \pm$)

$$\{f_{1\,2}^{\epsilon\,\epsilon}, f_{2\,3}^{\epsilon\,\epsilon}, f_{3\,1}^{\epsilon\,\epsilon} \quad =: \quad f^{\epsilon\,\epsilon}\{2(x\,y - y\,z - x\,z) + x\,y\,z,\ \textit{cyclic perm.}\}$$

with

$$f^{\epsilon\,\epsilon} = Exp[-\frac{1}{2}(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})](x\,y\,z)^{-3/4}(x-y)^{-1/8}(x-z)^{-1/8}(y-z)^{-1/8}$$
$$\left(C_1(x-z)^{-1/2} + \epsilon\, C_2(y-z)^{-1/2}\right)$$

$$\{f_{[1\,2]}^{+-}, f_{[2\,3]}^{+-}, f_{[3\,1]}^{+-}\} = f^{+-}\{2(x\,y - y\,z - x\,z) + x\,y\,z,\ \textit{cyclic perm.}\}$$

$$f^{+-} = C_3\, Exp[-\frac{1}{2}(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})]\,(x\,y\,z)^{-3/4}\,((x-y)(y-z)(z-x))^{-1/8}\,(x-y)^{-1/2}$$

- Level 2 : The six modes $f^{+-}_{(k\,l)} =: k_{k\,l}$ are propagating modes

They obey Maxwell-like equations :

$$\begin{aligned} \delta k \sim 0 \quad &: \quad \frac{i}{2}\partial^p k_{pa} + \phi^p\, k_{pa} - 2\,\rho^{pq}\, k_{pq} = 0 \\ d\,k \sim 0 \quad &: \quad \frac{i}{2}\partial_{[a} k_{b]c} - \phi_{[a}\, k_{b]c} + \mu_{ab}{}^p\, k_{pc} + 2\,\rho_{[a|c|}{}^p\, k_{b]p} = 0 \end{aligned}$$

Their compatibility is guaranteed by “Bianchi” identities : $d^2 = 0 = \delta^2$.
These equations split into :
• five constraint equations (no “time derivative”)
• six evolution equations (with respect to the “time” $\beta^0 = \beta^1 + \beta^2 + \beta^3$

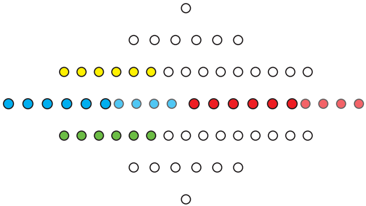
The general solution is parametrized by two arbitrary functions of two variables (leaving in a plane $\beta^0 = Cte$) from which we compute (via an Euler-Darboux-Poisson equation) the Cauchy data for the six k_{ab}, which then propagate thanks to the six evolution equations : $\partial_0 k_{ab} = \dots$.

They also may be described as modes superposition: far from the walls (in terms of plane waves), or when bouncing on a wall, far from the corners (in terms of special functions of Legendre or Kummer).

- Level 3 : A similar analysis can be done. The 20 components of $\Psi = \frac{1}{\sqrt{2}} \sum_\epsilon \frac{1}{3!} f^\epsilon \eta_{p\,q\,r} \tilde{b}^p_\epsilon \tilde{b}^q_\epsilon \tilde{b}^r_\epsilon + \frac{1}{2} h^\epsilon_{p\,q,r} \tilde{b}^p_{-\epsilon} \tilde{b}^q_{-\epsilon} \tilde{b}^r_\epsilon$ split into 10 +10 that decouple. Defining the dual components $h^\epsilon{}_{a\,b} = \frac{1}{2} \eta_a{}^{p\,q} h^\epsilon_{p\,q,b}$, all modes may be expressed in terms of $h^\epsilon{}_{(a\,b)}$.

Hilbert Space Structure

The solution space diamond :

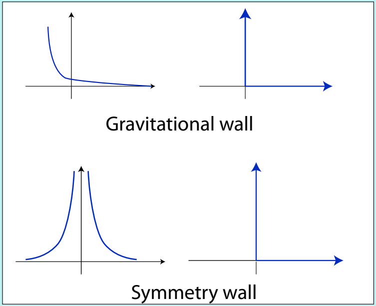


C_F	*dim*
3	1
2	6
1	15
0	20=(3+1+6)+(3+1+6)
-1	15=6+3+3+3
-2	6=3+3
-3	1

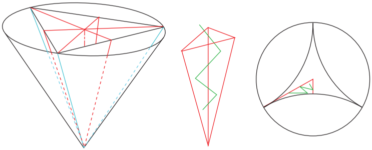
$$\mathcal{V}^{(0)} = V_1^{(0)}, \quad \mathcal{V}^{(1)} = V_2^{(1)}, \quad \mathcal{V}^{(2)} = V_3^{(2)} \oplus V_{1,\infty^2}^{(2)}, \quad \mathcal{V}^{(3)} = V_{2,\infty^2}^{(3)} \oplus V_{2,\infty^2}^{(2)}$$

Sharp wall approximation : $\exp[-2\,w(\beta)] \mapsto \infty\theta[-w(\beta)]$

$$\coth^2[w(\beta)] \mapsto \delta[w(\beta)] \sim \infty\theta[-w(\beta)]$$



The spinorial wave function of the Universe $\Psi(\beta^a)$ propagates within the (various) Weyl chamber(s) and "reflects" on the walls (= simple roots of AE_3). In the small-wavelength limit, the "reflection operators" define a *spinorial extension of the Weyl group of* AE_3 (Damour, Hillmann '09) defined within some subspaces of $\mathrm{Spin}(8,4)$



$$\widehat{\mathcal{R}}_{\alpha_i} = \exp\left(-i\,\frac{\pi}{2}\,\widehat{\varepsilon}_{\alpha_i}\widehat{J}_{\alpha_i}\right)$$

with $\widehat{J}_{\alpha_i} = \{\widehat{S}_{23}, \widehat{S}_{31}, \widehat{J}_{11}\}$ and $\widehat{\varepsilon}^2_{\alpha_i} = \mathrm{Id}$

THE "SQUARED-MASS" QUARTIC OPERATOR $\widehat{\mu}^2$ IN $\widehat{H}$

In the middle of the Weyl chamber (far from all the hyperplanes $\alpha_i(\beta) = 0$):

$$2\,\widehat{H} \simeq \widehat{\pi}^2 + \widehat{\mu}^2$$

where $\widehat{\mu}^2 \sim \sum \widehat{\Phi}^4$ gathers many complicated quartic-in-fermions terms (including $\sum \widehat{S}_{ab}^2$ and the infamous ψ^4 terms of supergravity).

Remarkable Kac-Moody-related facts:

- $\widehat{\mu}^2 \in$ *Center* of the algebra generated by the $K(AE_3)$ generators $\widehat{S}_{ab}$, $\widehat{J}_{ab}$
- $\widehat{\mu}^2$ is $\sim$ the square of a very simple operator $\in$ Center

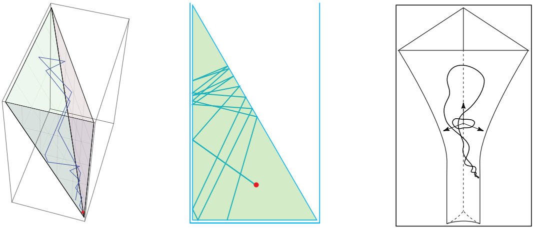
$$\widehat{\mu}^2 = \frac{1}{2} - \frac{7}{8}\,\widehat{C}_F^2$$

where $\widehat{C}_F := \frac{1}{2}\, G_{ab}\,\overline{\widehat{\Phi}}^a\,\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^b$.

The "squared-mass" Quartic Operator $\widehat{\mu}^2$ in $\widehat{H}$

$$\widehat{\mu}^2 = \frac{1}{2} - \frac{7}{8}\,\widehat{C}_F^2 = -\frac{59}{8}, -3, -\frac{3}{2}, +\frac{1}{2}$$

Spacelike trajectories : bounce before reaching the singularity!



CONCLUSIONS

• The case studied of the quantum dynamics of a triaxially squashed 3-sphere (Bianchi IX model) in (simple, $D = 4$) supergravity confirms the hidden presence of hyperbolic Kac-Moody structures in supergravity. [Here, AE_3 and $K(AE_3)$]

• The wave function of the Universe $\Psi(\beta^1, \beta^2, \beta^3)$ is a 64-dimensional spinor of $\mathrm{Spin}(8,4)$ which satisfies Dirac-like, and Klein-Gordon-like, wave equations describing the propagation of a "quantum spinning particle" reflecting off spin-dependent potential walls which are built from quantum operators $\widehat{S}_{12}, \widehat{S}_{23}, \widehat{S}_{31}, \widehat{J}_{11}, \widehat{J}_{22}, \widehat{J}_{33}$ that generate a 64-dim representation of $K(AE_3)$. The squared-mass term $\widehat{\mu}^2$ in the KG equation belongs to the center of this algebra.

• This result might help in clarifying the extent to which the gravity/coset correspondence holds (here for the coset $AE_3/K(AE_3)$, and more interestingly for $E_{10}/K(E_{10})$).

Conclusions

We recover, in the framework of simple (D=4) supergravity, elements of the hidden hyperbolic Kac–Moody structures.

Our dynamical model differs significantly from those obtained in previous works. We obtain modes for all values of N_F (all previous works agreed on the inexistence of solutions for odd values of N_F).

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In the short wave limit (wave packets), some components of the wave function of the Universe behave like tachyonic particle. They may "bounce near the past", and thus escape from the singularity.