Reprinted from ANNALS OF PHYSICS
All Rights Reserved by Academic Press, New York and London

Vol. 76, No. 2, April 1973
Printed in Belgium

Dynamics of Quantum Correlations

O. KÜBLER AND H. D. ZEH

Institut für Theoretische Physik der Universität Heidelberg, 69, Heidelberg, Germany

Received January 24, 1972

The density matrix describing the state of a subsystem of a physical system whose time dependence is assumed to follow a Schrödinger equation does not itself obey a von Neumann equation. The behavior of the eigenvalues and eigenfunctions of this density matrix is studied. An expression for the rate at which initially separating systems are de-separated is derived in perturbation theory. Indications are given that coherent photon states are more stable in the presence of charged particles than photon number eigenstates. The possible dynamical origin of super selection rules is discussed. A simple model is solved analytically.

1. INTRODUCTION

The general validity of the main axiom of quantum theory, i.e., of the superposition principle, has been often questioned. It has been emphasized that certain kinds of superpositions never have been observed [1]. Examples are superpositions of states which differ by their charge quantum number, or which have integer and half-integer spin. The open interpretation of superpositions of "macroscopically different states" [2] is apparently a related problem.

Therefore it has been suggested [1, 3, 4] to restrict the superposition principle "by postulate." This modified principle either forbids superpositions of states from certain different subspaces of the underlying Hilbert space, or it restricts the class of observables to those connected with operators that commute with projectors on these subspaces ("super selection rules"), so that the corresponding relative phases cannot be observed.

On the other hand, there are dynamical reasons—without adding new postulates—for certain phase relations of most physical systems being absent. This situation occurs because in general there are "quantum correlations" between any two systems in the sense that their total wave function (if it exists itself) does not separate in the form $\psi = \psi_1 \psi_2$. As a consequence all observations of, say, system one, are described by a density matrix $\rho^{(1)} = \mathrm{Tr}_2\, \psi\psi^*$ which does not correspond to a pure state. This density matrix may be diagonalized and hence be represented by an ensemble of orthogonal states with associated probabilities. The "most probable type" (see below) of these states will depend on the dynamics, i.e., on the interaction between the two systems. Since no phase relations between these states are defined, as far as only one system is considered, the problem of superselection rules may be a dynamical one. This question is to be discussed in the following.

The relative phases still could be observed by correlation experiments, i.e., by measurements of the total system. However, all "total systems" on which measurements may be performed are subsystems of the universe. The reason for this is a principal problem of observation theory as has been pointed out elsewhere [5]. These subsystems interact with the "remainder of the universe", and therefore must be in "mixed states." The interaction is the more effective, the more macroscopic the system. The "effectivity" of the interaction, i.e., its ability to correlate the wave functions, hence may be the reason that certain properties of physical systems behave in the way which we call macroscopic.

2. DYNAMICAL DE-SEPARATION IN LOWEST ORDER OF TIME

Any vector ψ contained in the tensor product of two Hilbert spaces $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ can be expanded in the Schmidt canonical form [6]

$$\psi = \sum_i p_i^{1/2} \varphi_i \Phi_i \,, \tag{1}$$

where $\{\varphi_i\}$ and $\{\Phi_i\}$ are certain orthonormal systems in $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. These systems may always be completed to form basis systems. From this form of the wave function, one may derive directly the diagonal form of the density matrices,

$$\rho^{(1)} = \sum_i | \varphi_i \rangle p_i \langle \varphi_i | \tag{2a}$$

$$\rho^{(2)} = \sum_i | \Phi_i \rangle p_i \langle \Phi_i | \,. \tag{2b}$$

The basis systems $\{\varphi_i\}$ and $\{\Phi_i\}$ are defined uniquely by the state ψ, except for degeneracy of the coefficients p_i.

If the vector ψ depends on the time, the coefficients p_i and the systems $\{\varphi_i\}$, $\{\Phi_i\}$ carry an independent time dependence. This has two interesting consequences [7]. (a) The density matrices $\rho^{(1)}$ and $\rho^{(2)}$ in general will not obey Schrödinger (von Neumann) equations of the kind $i\hbar\dot{\rho}^{(1)} = [H^{(1)}, \rho^{(1)}]$. This can be seen easily since for each component φ_i there would be another Hamiltonian

$$H_i^{(1)} = \mathrm{Tr}_2\{H \Phi_i \Phi_i^*\}.$$

(b) The "entropy" of the subsystems $S^{(1)}$ $S^{(2)}$ $\sum_i p_i \ln p_i$ will vary with time.

Although there is a linear equation of motion, the components of Eq. (1) do not behave independently. The rate of change of the numbers p_i as well as that of the states φ_i and Φ_i depends on the total present state.

The case of initially separating systems

$$\psi(0) = \varphi(0)\, \Phi(0) = \varphi_0 \Phi_0 \tag{3}$$

is of particular interest. An arbitrary time dependence $\psi(t)$ does not necessarily destroy the separation. Which dynamical properties are responsible for a deseparation? This question is related to that of stability of phase relations.[1]

If the Hamiltonian is constant, one has in second order of time

$$\begin{aligned} \psi(t) = \psi(0) - it\hbar^{-1} \sum_{j,m} \langle jm \mid H \mid 00 \rangle \varphi_j \Phi_m \\ - \frac{t^2}{2} \hbar^{-2} \sum_{j,m} \langle jm \mid H^2 \mid 00 \rangle \varphi_j \Phi_m =: \sum_j \varphi_j \Phi^{(j)}(t). \end{aligned} \tag{4}$$

The second equation defines the "relative states" $\Phi^{(j)}(t)$,

$$\Phi^{(j)}(t) = \Phi_0 \delta_{j0} - it\hbar^{-1} \sum_m \langle jm \mid H \mid 00 \rangle \Phi_m - t^2\hbar^{-2} \sum_m \langle jm \mid H^2 \mid 00 \rangle \Phi_m \,. \tag{5}$$

They are not orthogonal:

$$\begin{aligned} \langle \Phi^{(j)} \mid \Phi^{(j')} \rangle = \delta_{j0}\delta_{jj'} \\ + it\hbar^{-1}[\delta_{j0}\langle j'0 \mid H \mid 00 \rangle - \delta_{j'0}\langle j0 \mid H \mid 00 \rangle] \\ + \frac{t^2}{2} \hbar^{-2} \Big[2 \sum_m \langle jm \mid H \mid 00 \rangle \langle 00 \mid H \mid j'm \rangle \\ - \delta_{j0}\langle j'0 \mid H^2 \mid 00 \rangle - \delta_{j'0}\langle j0 \mid H^2 \mid 00 \rangle \Big]. \end{aligned} \tag{6}$$

Equation (6) may be diagonalized by a unitary matrix

$$\sum_{j,j'} u_{ij} \langle \Phi^{(j)} \mid \Phi^{(j')} \rangle u_{j'i'}^* = p_i \delta_{ii'} \tag{7}$$

---

[1] It appears worthy to emphasize that the representation by density matrices [Eq. (2)] and hence any disappearance of phase relations in the sense discussed here are based on the probability interpretation of quantum mechanics and therefore should not be used to explain the latter. The phrase that a member of the ensembles represented by Eq. (2) can be "picked out" by the mere increase of information would refer to classical concepts of observation (see Refs. 5 and 7). The collapse of the wave function certainly can not be *derived* from the formal decomposition (1).

This matrix defines a unitary operator in $\mathscr{H}_1$,

$$U = \sum_{i,j} | \varphi_i \rangle u_{ij} \langle \varphi_j | \tag{8a}$$

and a non-unitary operator in $\mathscr{H}_2$,

$$T = \sum_{i,j} | \Phi^{(i)} \rangle u^*_{ji} \langle \Phi^{(j)} | . \tag{8b}$$

The transformation

$$\hat{\varphi}_i(t) = \sum_j u_{ij}(t)\, \varphi_j \tag{9a}$$

$$p_i^{1/2}(t)\, \hat{\Phi}_i(t) = \sum_j u^*_{ji}(t)\, \Phi^{(j)}(t) \tag{9b}$$

leads to the Schmidt canonical form.

The eigenvalues $p_i(t)$ are solutions of the equation

$$\det[p_j \delta_{jj'} - \langle \Phi^{(j)} | \Phi^{(j')} \rangle] = 0. \tag{10}$$

Expanding the determinant by the zeroth column and row, one obtains up to quadratic order of t,

$$p_i^N - p_i^{N-1} + t^2 \hbar^{-2} p_i^{N-2} \sum_{j \neq 0, m \neq 0} |\langle jm | H | 00 \rangle|^2 = 0, \tag{11}$$

where N is a cut-off rank of the matrix. Roots are

$$p_0 = 1 - t^2 \hbar^{-2} \sum_{j \neq 0, m \neq 0} |\langle jm | H | 00 \rangle|^2 \tag{12a}$$

$$p_1 = t^2 \hbar^{-2} \sum_{j \neq 0, m \neq 0} |\langle jm | H | 00 \rangle|^2 \tag{12b}$$

and $N - 2$ vanishing roots. The result converges for $N \to \infty$ if $\sum_{j,m} |\langle jm | H | 00 \rangle|^2$ does.

The quantity

$$A = \sum_{j \neq 0, m \neq 0} |\langle jm | H | 00 \rangle|^2$$

measures the rate of dynamical de-separation in lowest order of time. It differs from the quantity

$$B = \sum_{j,m \neq 0,0} |\langle jm | H | 00 \rangle|^2 ,$$

which measures the total change of the state, i.e., the rate at which $\psi(t)$ becomes orthogonal to $\psi(0)$, by the terms

$$B - A = \sum_{j \neq 0} |\langle j0 | H | 00 \rangle|^2 + \sum_{m \neq 0} |\langle 0m | H | 00 \rangle|^2. \tag{13}$$

In the general case of initially "mixed states" the eigenvalues $p_i(t)$ vary in first order of time. The time derivative is obtained by diagonalizing Eq. (10) in first order of perturbation theory,

$$\hbar \frac{dp_i^{1/2}(t)}{dt} = \operatorname{Im} \sum_j p_j^{1/2}(t) \langle ii | H | jj \rangle, \tag{14}$$

where the states $| ii \rangle$ refer to the Schmidt canonical decomposition at time t.

The time dependence of the basic states is given by (neglecting degeneracy of probabilities)

$$i\hbar \frac{d\varphi_i}{dt} = \sum_{j \neq i} (p_i - p_j)^{-1} \sum_m p_m^{1/2} [p_i^{1/2} \langle ji | H | mm \rangle - p_j^{1/2} \langle mm | H | ij \rangle]\, \varphi_j \tag{14'}$$

$$i\hbar \frac{d\Phi_i}{dt} = \sum_{j \neq i} (p_i - p_j)^{-1} \sum_m p_m^{1/2} [p_i^{1/2} \langle ij | H | mm \rangle - p_j^{1/2} \langle mm | H | ji \rangle]\, \Phi_j$$

$$+ p_i^{-1/2} \operatorname{Re} \sum_m p_m^{1/2} \langle ii | H | mm \rangle\, \Phi_i . \tag{14''}$$

The asymmetry is due to phase ambiguity.

Equations (14′) and (14″) show a remarkable "probability resonance". It gives rise to a rapid correlated variation of two almost degenerate members of the expansion (1).

3. Coupled Harmonic Oscillators

Consider the Hamiltonian

$$H = H_0 + W \tag{15}$$

$$H_0 = \hbar\omega(a_1^+ a_1 + a_2^+ a_2) \tag{15a}$$

$$W = \epsilon\hbar\omega(a_1^+ + a_1)(a_2^+ + a_2). \tag{15b}$$

The rate of *de*-separation depends critically on the initial wave functions $\varphi(0)$ and $\phi(0)$.

If the subsystems are initially in eigenstates of H_0, $\varphi(0) = \varphi_n$, $\Phi(0) = \Phi_N$, where $H_0 \varphi_n \Phi_N = \hbar\omega(N+n)\, \varphi_n \Phi_N$, H_0 contributes neither to A nor to B. One obtains

$$A = B = \epsilon^2 \hbar^2 \omega^2 (4Nn + 2N + 2n + 1). \tag{16}$$

The left equation means that the interaction is "fully effective" in deseparating the systems. The result shows that A becomes large even for small ϵ in the "macroscopic limit" of large n and N.

If the subsystems are initially in coherent states [8], $\varphi(0) = \varphi_{\alpha_1}$, $\Phi(0) = \Phi_{\alpha_2}$, where $a_1 a_2 \varphi_{\alpha_1} \Phi_{\alpha_2} = \alpha_1 \alpha_2 \varphi_{\alpha_1} \Phi_{\alpha_2}$, H_0 contributes to the measure of time dependence B. Since there is no simple orthonormal set that contains a coherent state as a member, it is convenient to introduce the projection operators

$$P_{\alpha_1} = P^{(1)}_{\alpha_1} \otimes \mathbf{1}^{(2)} \tag{17a}$$

$$P_{\alpha_1} = \mathbf{1}^{(1)} \otimes P^{(2)}_{\alpha_2}. \tag{17b}$$

$P^{(1)}_{\alpha_1}$ and $P^{(2)}_{\alpha_2}$ are the projectors on φ_{α_1} and Φ_{α_2}, respectively. The measure of *de*-separation may be then rewritten

$$A = \langle \alpha_1 \alpha_2 \mid W(\mathbf{1} - P_{\alpha_1})(\mathbf{1} - P_{\alpha_2})\, W \mid \alpha_1 \alpha_2 \rangle. \tag{18}$$

By using the above definition of coherent states and the commutation relations, one obtains

$$A = \epsilon^2 \hbar^2 \omega^2. \tag{19}$$

In the case of coherent states, the measure of deseparation remains small for small ϵ even in the macroscopic limit (large α). Although the problem should certainly be investigated beyond perturbation theory, the results are expected to indicate why macroscopic oscillators are usually observed in coherent states, in contrast to microscopic ones which are observed in energy eigenstates (due to interactions with the electromagnetic field).

Similar results are obtained for a fermion–boson interaction

$$\begin{aligned} W &= \int d^3q\, d^3q' \{ f(\mathbf{q}, \mathbf{q}')\, a_{\mathbf{q}}^{+} b_{\mathbf{q}'}^{+} b_{\mathbf{q}+\mathbf{q}'} + \text{herm. conj.} \} \\ &=: W_1 + W_1^{+} \end{aligned} \tag{20}$$

with a cut-off function $f(\mathbf{q}, \mathbf{q}')$; $a_{\mathbf{q}}^{+}$ creates a boson and $b_{\mathbf{q}'}^{+}$ a fermion.

If the initial state consists of a fermion wave packet and bosons in coherent states,

$$\mid \varphi(\mathbf{p}); \{\alpha_{\mathbf{k}}\} \rangle = \int d^3p\, \varphi(\mathbf{p})\, b_{\mathbf{p}}^{+} \mid 0; \{\alpha_{\mathbf{k}}\} \rangle \tag{21}$$

$$a_{\mathbf{k}} \mid 0; \{\alpha_{\mathbf{k}}\} \rangle = \alpha_{\mathbf{k}} \mid 0, \{\alpha_{\mathbf{k}}\} \rangle \ \forall \mathbf{k} \tag{21'}$$

one obtains for the measure of *de*-separation

$$A = \langle \varphi_i\{\alpha_{\mathbf{k}}\} \mid W_1^{+}(\mathbf{1} - P_\varphi)(\mathbf{1} - P_{\{\alpha_{\mathbf{k}}\}})\, W_1 \mid \varphi_i\{\alpha_{\mathbf{k}}\} \rangle \qquad (2$$

$$W_1 \mid \varphi_i\{\alpha_{\mathbf{k}}\} \rangle = \int d^3q\, d^3q'\, f(\mathbf{q}, \mathbf{q}')\, \varphi(\mathbf{q} + \mathbf{q}')\, a_{\mathbf{q}}^{+} b_{\mathbf{q}'}^{+} \mid 0_i\{\alpha_{\mathbf{k}}\} \rangle \qquad (22$$

$$\begin{aligned} A = \int d^3q \Big[&\int d^3q' \mid \varphi(\mathbf{q} + \mathbf{q}') \mid^2 \mid f(\mathbf{q}, \mathbf{q}') \mid^2 \\ &- \Big| \int d^3q'\, f(\mathbf{q}, \mathbf{q}')\, \varphi(\mathbf{q} + \mathbf{q}')\, \varphi^*(\mathbf{q}') \Big|^2 \Big]. \end{aligned} \qquad (22$$

This result is independent of $\{\alpha_{\mathbf{k}}\}$. It holds for the boson vacuum $\alpha_{\mathbf{k}} = 0$, $\forall$ and therefore must be entirely due to renormalization effects.

Additional terms occur for other than coherent states. The rate of *de*-separati becomes for occupation number eigenstates $\mid \varphi(\mathbf{p}), \{n_{\mathbf{q}}\} \rangle$

$$\begin{aligned} A = \int d^3q\, (n_{\mathbf{q}} + 1) \Big[&\int d^3q' \mid f(\mathbf{q}, \mathbf{q}') \mid^2 \mid \varphi(\mathbf{q} + \mathbf{q}') \mid^2 \\ &- \Big| \int d^3q'\, f(\mathbf{q}, \mathbf{q}')\, \varphi(\mathbf{q} + \mathbf{q}')\, \varphi^*(\mathbf{q}') \Big|^2 \Big] \\ + \int d^3q\, n_{\mathbf{q}} \Big[&\int d^3q' \mid f(\mathbf{q}, \mathbf{q}') \mid^2 \mid \varphi(\mathbf{q}') \mid^2 - \Big| \int d^3q'\, f(\mathbf{q}, \mathbf{q}')\, \varphi(\mathbf{q} + \mathbf{q}')\, \varphi^*(\mathbf{q}') \Big|^2 \Big] \end{aligned} \qquad (2$$

The terms which add to the renormalization terms are positive except f $n_{\mathbf{q}} = 0$, $\forall \mathbf{q}$. Occupation number eigenstates are therefore unstable in the presen of charged particles, and this is expected to be the reason that photons are usua observed as "classical" fields [8]. This result together with the interpretation quantum theory presented in Ref. 5 and 7 seems to come very close to Schrödinge original intentions.

4. DESTRUCTION OF PHASES BETWEEN STATES OF DIFFERENT CHARGE

Superselection rules have been postulated particularly for states of differe charge. It appears that phase relations between such states—if they should ex at some instant—will be destroyed rapidly by the different interaction with enviro ment.

Consider the superposition of a proton and a neutron, $a\chi_p + b\chi_n$, interacti with and initially decoupled from its environment represented by a wave functi

Φ. For the present purpose the environment, will be assumed to consist of a large number N of independent charged oscillators,

$$\Phi = \prod_{i=1}^{N} \varphi_i \, .$$

The wave functions of the oscillators will then be perturbed by the Coulomb field of the proton.

$$(a\chi_p + b\chi_n) \prod_{i=1}^{N} \varphi_i \to a\chi_p \prod_{i=1}^{N} \varphi_i' + b\chi_n \prod_{i=1}^{N} \varphi_i \, . \tag{24}$$

As a consequence of the large number of particles N, the two states of the environment Φ and Φ' coupled to the proton or neutron, respectively, are almost orthogonal,

$$\langle \Phi \mid \Phi' \rangle = \prod_{i=1}^{N} \langle \varphi_i \mid \varphi_i' \rangle \approx 0,$$

even if each oscillator is perturbed only very slightly. Perturbation theory therefore appears justified in order to calculate the rate of deseparation. Assuming $N = 10^{20}$ oscillators of charge $Z = 20$ and a mean oscillator radius $\langle r_i^2 \rangle^{1/2} = 10^{-11}$ cm at mean distance $\bar{R} = 1$ cm, one obtains the rate of *de*-separation $A = \hbar^2 (10^{-8}\,\text{sec})^{-2}$.

Similarly one may ask for the maximum distance l at which a charged particle may interfere with itself. In this case the interactions produced by the two positions of the charge differ by a dipole moment of magnitude el. Assuming the same environment as in the case of different charges, the above value of A has to be multiplied by a factor $(l/\bar{R})^2$. The resulting time scales do not contradict experiments concerned with the Aharonov–Bohm effect. [9] On the other hand, they indicate for example that a spherical wave of a charged particle will be decomposed very soon into non-interferring wave packets.

Therefore, it is perhaps remarkable that the decomposing states $\{\varphi_i(\mathbf{x}, t)\}$ and $\{\Phi_i(\mathbf{y}, t)\}$ of two correlated fields $\psi(\mathbf{x}, \mathbf{y}, t)$—compare Eq. (1)—show some similarity to "quanta" (fermions): They (a) do not interfere with one another, (b) are mutually orthogonal, and (c) can appear and disappear pairwise. Needless to say, such an identification would require a complete *re*-interpretation of quantum theory.

It seems that we call properties macroscopic (e.g., the charge) if an unavoidable interaction with the environment always induces a Schmidt canonical decomposition which separates the subspaces corresponding to these properties, whereas we call them microscopic (e.g., the spin) if we can select the environment (construct a measuring apparatus) in such a way that the decomposition creates superpositions of these subspaces. The set of those Hermitian operators which correspond to observables therefore appear to be determined by the dynamics and must not be chosen ad hoc.

5. AN EXACTLY SOLUBLE MODEL

A simple minded model is investigated, which on the one hand serves further to illustrate the arguments raised in section 2, and, on the other hand, exhibits interesting features pointing beyond the results of perturbation theory.

5.1. *Definition of the Model*

The model is defined by placing two particles with two degrees of freedom each, endowed with magnetic momenta $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, respectively, in a static homogeneous magnetic field $\mathscr{H}$ (defining a z-direction in space). The two systems are coupled by an interaction energy of the type $\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2$.

The Hamiltonian of the total system is given by:

$$H = H_0 + W \qquad \text{(25}$$

$$H_0 = (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) \cdot \mathscr{H} = (\mu_1 \sigma_z^{(1)} + \mu_2 \sigma_z^{(2)}) \mathscr{H} \qquad \text{(25a}$$

$$W = \lambda \boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2 = \lambda \mu_1 \mu_2 (\sigma_x^{(1)} \sigma_x^{(2)} + \sigma_y^{(1)} \sigma_y^{(2)} + \sigma_z^{(1)} \sigma_z^{(2)}), \qquad \text{(25b}$$

where λ, μ_1, μ_2, are constants and σ_x, σ_y, σ_z are the Pauli matrices.

The eigenvalues and eigenstates of H are given by

$$\begin{aligned}
E_1 &= \mu + k & \mid E_1 \rangle &= \mid 1\tilde{1} \rangle \\
E_2 &= -\mu + n & \mid E_2 \rangle &= N^{-1}\{(h + n) \mid 2\tilde{1} \rangle + 2\mu \mid 1\tilde{2} \rangle\} \qquad \text{(26} \\
E_3 &= -\mu - n & \mid E_3 \rangle &= N^{-1}\{-2\mu \mid 2\tilde{1} \rangle + (h + n) \mid 1\tilde{2} \rangle\} \\
E_4 &= \mu - k & \mid E_4 \rangle &= \mid 2\tilde{2} \rangle
\end{aligned}$$

where

$$\begin{aligned}
k &= \mathscr{H}(\mu_1 + \mu_2) & n &= (h^2 + \overset{4\mu^2}{\cancel{k^2}})^{1/2} \\
h &= \mathscr{H}(\mu_2 - \mu_1) & N^2 &= 2n(h + n) \\
\mu &= \lambda \mu_1 \mu_2 & & \\
\mid i\tilde{k} \rangle &= \mid i \rangle \otimes \mid \tilde{k} \rangle & & \\
\sigma_z^{(1)} \mid 1 \rangle &= \mid 1 \rangle & \sigma_z^{(2)} \mid \tilde{1} \rangle &= \mid \tilde{1} \rangle \\
\sigma_z^{(2)} \mid 2 \rangle &= - \mid 2 \rangle & \sigma_z^{(2)} \mid \tilde{2} \rangle &= - \mid \tilde{2} \rangle .
\end{aligned}$$

The density matrix of the total system becomes (in all that follows, we choose $h = 1$)

$$\rho(t) = | \psi(t)\rangle\langle\psi(t)| = \sum_{k,k'=1}^{2} \psi_{k'}^{*}\psi_k \,| E_k\rangle\langle E_{k'} | \exp\{i\,(E_{k'} - E_k)\,t\}. \tag{27}$$

5.2. *The Density Matrix of One Subsystem*

By performing the trace over subsystem 2, we obtain the density matrix of subsystem 1

$$\rho^{(1)}(t) = \mathrm{Tr}_2\,\rho(t) = \sum_{i,k=1}^{2} \rho_{ik}^{(1)}(t)| i\rangle\langle k \,| , \tag{28}$$

where

$$\rho_{11} = | \psi_1 |^2 + \frac{1}{2}\,[| \psi_2 |^2 + | \psi_3 |^2] + \frac{h}{2}\,n^{-1}[| \psi_3 |^2 - | \psi_2 |^2] + 2\mu n^{-1}\,\mathrm{Re}\,\psi_2\psi_3{}^*\exp\{-2itn\} \tag{28a}$$

$$\begin{aligned}\rho_{12} = \rho_{21}^{*} = {} & \exp\{ikt\}\,N^{-1}\{(h+n)\exp\{itn\}[\psi_2{}^*\psi_1\exp\{-2i\mu t\} \\ & + \psi_4{}^*\psi_3\exp\{2i\mu t\}] \\ & + 2\mu\exp\{-itn\}[\psi_4{}^*\psi_2\exp\{2i\mu t\} \\ & - \psi_3{}^*\psi_2\exp\{-2i\mu t\}]\}\end{aligned} \tag{28b}$$

$$\rho_{22} = 1 - \rho_{11}\,. \tag{28c}$$

The time dependence of $\rho^{(1)}(t)$ contains in general all frequencies of the total system. The coupled quantum system shares this property with a coupled classical system where a (time dependent) Hamiltonian $H^{(1)}$ exists.

The question when $\rho^{(1)}(t)$ corresponds to a pure state and when it does not is treated by diagonalizing $\rho^{(1)}$ and looking at the eigenvalues, which are given by

$$\begin{aligned} p_{>}(t) &= \tfrac{1}{2} + \tfrac{1}{2}R^{1/2}(t) \\ p_{<}(t) &= \tfrac{1}{2} - \tfrac{1}{2}R^{1/2}(t), \end{aligned} \tag{29}$$

where

$$R(t) = (\rho_{11} - \rho_{22})^2 + 4\,| \rho_{12} |^2. \tag{30}$$

We have to discuss the range of values that R may assume. This is done most easily by introducing:

$$\begin{aligned}
hn^{-1} &= \sin\gamma & 2\mu n^{-1} &= \cos\gamma, & &\text{where}\quad 0 \leqslant \gamma \leqslant 2\pi \\
\psi_k &= S_k\exp\{i\varphi_{k'}\} & k &= 1,\dots,4 & & \\
S_1 &= \cos\alpha_1\cos\alpha_3 & S_2 &= \sin\alpha_1\sin\alpha_3\,, & &\text{where}\quad 0 \leqslant \alpha_k \leqslant \pi/2 \\
S_4 &= \cos\alpha_1\sin\alpha_3 & S_3 &= \sin\alpha_1\cos\alpha_2 & & \qquad\qquad (31) \\
\Omega_1 &= 4\mu & \Omega_2 &= 2\mathrm{n} & & \\
\Phi_1 &= \varphi_2 + \varphi_3 - \varphi_1 - \varphi_4 & \Phi_2 &= \varphi_3 - \varphi_2\,. & &
\end{aligned}$$

With these notations we obtain

$$R = 2\cos^2\alpha_1\sin^2\alpha_1\{1 - B(t)\sin 2\alpha_3\} + \cos^4\alpha_1\cos^2 2\alpha_3 + a^2(t)\sin^4\alpha_1 \tag{30'}$$

$$\begin{aligned} &= 1 - \{(1 - a^2(t))^{1/2}\sin^2\alpha_1 - \sin 2\alpha_3\cos^2\alpha_1\}^{1/2} \\ &\quad - 2\sin^2\alpha_1\cos^2\alpha_1\sin 2\alpha_3\{(1 - a^2(t))^{1/2} - B(t)\}, \end{aligned} \tag{30''}$$

where

$$B(t) = b(t)\cos(\Omega_1 t + \Phi_1) - \cos\gamma\sin(\Omega_1 t + \Phi_2)\sin(\Omega_2 t + \Phi_2) \tag{30a}$$

$$b(t) = -\sin\gamma\sin 2\alpha_2 + \cos\gamma\cos 2\alpha_2\cos(\Omega_2 t + \Phi_2) \tag{30b}$$

$$a(t) = \sin\gamma\cos 2\alpha_2 + \cos\gamma\sin 2\alpha_2\cos(\Omega_2 t + \Phi_2). \tag{30c}$$

Formula (30′) is useful for establishing a lower bound for R while (30″) will be employed to discuss the maximum values of R.

The range of $a(t)$ is given by

$$\begin{aligned}\underline{a} &= \min\{\sin(\gamma + 2\alpha_2), \sin(\gamma - 2\alpha_2)\} \leqslant a(t) \leqslant \max\{\sin(\gamma + 2\alpha_2)_1 \sin(\gamma - 2\alpha_2)\} \\ &= \bar{a}. \end{aligned} \qquad (32$$

Furthermore we get

$$B^2(t) = 1 - a^2(t) - c^2(t), \qquad (33$$

where

$$c(t) = b(t)\sin(\Omega_1 t + \Phi_1) + \cos\gamma\cos(\Omega_2 t + \Phi_1)\sin(\Omega_2 t + \Phi_2).$$

c range of values of R is thus bounded from below, we obtain

$$R \geqslant 2\cos^2\alpha_1\sin^2\alpha_1\{1 - |\sin 2\alpha_3|\} + \cos^4\alpha_1\cos^2 2\alpha_3$$
$$+ \begin{cases} 0 & \text{if } \underline{a} \leqslant 0 \\ \underline{a}^2\sin^4\alpha_1 & \text{otherwise.} \end{cases} \tag{34}$$

ı upper bound for R is known from the outset, $R \leqslant 1$, since the diagonal ments of the density matrix obey the condition $\rho_{kk} \geqslant 0$. We have to investigate ıether $R = 1$ is always possible at some instant(s) t_1.

Because of (33), $R = 1$ is possible only when the following two conditions are nultaneously obeyed

$$1 - a^2(t_1) = \cot^4\alpha_1\sin^2 2\alpha_3 \tag{35a}$$
$$c(t_1) = 0. \tag{35b}$$

om (35a) we learn that only a subset $\{|\psi_0\rangle_{\text{pure}}\}$ of the total set of states $\{|\psi_0\rangle\}$ ıs a chance of yielding a density matrix $\rho^{(1)}(t)$ that corresponds to a pure state of e subsystem at some instant(s) t_1 as time evolves.

Consider a state from this subset. Assume (35a) solved. Now distinguish two ses:

(a) If the ratio Ω_1/Ω_2 is irrational, (35b) can be solved with arbitrary precision r sufficiently large times, though in general not exactly

(b) If the ratio Ω_1/Ω_2 is rational, pure states can occur only in accidental cases orresponding to a set of states of measure zero.)

For comparison with section 2, consider the subset of states

$$\{|\psi_0\rangle_{\text{sep}}\} \subset \{|\psi_0\rangle_{\text{pure}}\}$$

escribing initially decoupled systems. Any such state can be written in the form:

$$|\psi_0\rangle_{\text{sep}} = A_1A_2\,|1\tilde{1}\rangle + B_1A_2\,|2\tilde{1}\rangle + A_1B_2\,|1\tilde{2}\rangle + B_1B_2\,|2\tilde{2}\rangle \tag{36}$$

ith

$$|A_1|^2 + |B_1|^2 = |A_2|^2 + |B_2|^2 = 1$$
$$\varphi = \arg A_2 - \arg B_2 - (\arg A_1 - \arg B_1)$$

The time evolution of the eigenvalues $p_>$ (resp. $p_<$) of the density matrix has een calculated by computer for a number of states and a number of ratios Ω_1/Ω_2. ypical results are shown in the figure.

The eigenvalue $p_>$ is found in the vicinity of unity quite often. This is a conseuence of the special initial conditions in connection with the permitted range of

---

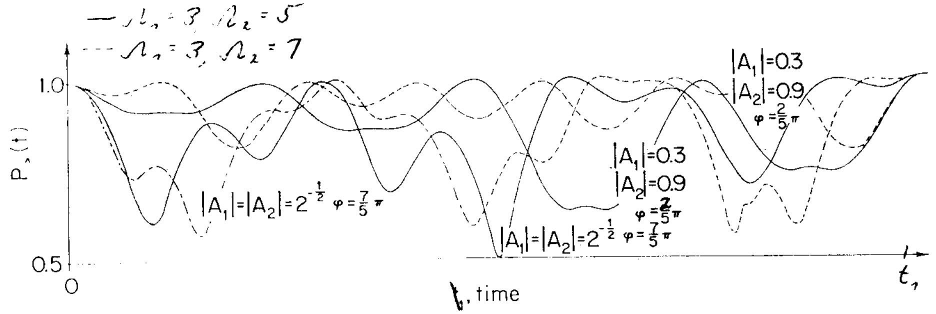

FIG. 1. Time evolution of the larger eigenvalue $p_>(t)$ of the density matrix $\rho^{(1)}(t) = \mathrm{Tr}_2\,\rho(t)$ of subsystem 1, say, of a total system composed of two interacting systems with two degrees of freedom each. Initially separating systems are considered. The temporal behavior of the eigenvalue is plotted for four different combinations of parameters [definitions see Eqs. (31) and (36)]. $p_> = 1$ corresponds to a pure state.

values derived for $R(t)$. This situation is probably different for a simple system in contact with a complex one (see Section 4) where a statistical distribution of eigenvalues is expected most of the time.

6. DISCUSSION

If the Schrödinger equation is assumed to be universally valid, physical systems can be in uncorrelated states only in exceptional cases. This holds even for weakly interacting systems since the universe is sufficiently old. Therefore in the conventional interpretation of quantum theory (i.e., when describing measurements phenomenologically), one should expect almost all systems to be represented by ensembles of many states whose phase relations are not defined.

The main problem arises: Why do microscopic systems usually occur in pure states? This question is obviously connected with the "collapse of the wave function" which is used in preparing pure states. The conventional phrase that this collapse describes the increase of information does not appear very satisfactory as it would mean that the process of information transfer is not a physical process like others (describable by a Schrödinger equation) [10].

REFERENCES

1. G. C. WICK, A. S. WIGHTMAN AND E. P. WIGNER, *Phys. Rev.* **88** (1952), 107.
2. A. Einstein's Letters No. 88 and 108 in "Briefwechsel zwischen A. Einstein und M Born" (M. Born, Ed.), München 1969; E. P. WIGNER, *Amer. J. Phys.* **31** (1963), 6.
3. R. F. STREATER AND A. S. WIGHTMAN, "TCP, Spin and Statistics, and all that," Benjamin Inc., New York, 1964.

4. J. M. Jauch, Foundations of Quantum Mechanics (Addison-Wesley, Reading, Mass. 1968); G. Ludwig, Lecture Notes in Physics No. 4, Springer, Berlin 1970; C. Piron, *Helv. Phys. Acta* **42** (1969), 330.
5. H. D. Zeh, *Found. Phys.* **1** (1970), 69.
6. E. Schmidt, *Math. Annalen* **63** (1907), 433; E. Schrödinger, *Proc. Cambridge Phil. Soc.* **31** (1935), 555; **32** (1936), 446.
7. H. D. Zeh, Enrico Fermi School of Physics, 49th Course (Varenna 1970).
8. R. J. Glauber, *Phys. Rev.* **131** (1963), 2766; P. Caruthers and M. M. Nieto, *Amer. J. Phys.* **33** (1965), 537.
9. G. Möllenstedt and H. Düker, *Z. Phys.* **145** (1956), 377; W. Bayh, *Z. Phys.* **169** (1962), 492.
10. H. D. Zeh, *Found. Phys.* **3** (1973), 109.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.