Figure 0.1

0.1 Coupled systems

0.1.1 The measurement problem

Elementary QM provides two *very* different rules for time evolution:

- Hamiltonian time evolution: $|\psi\rangle \mapsto e^{\frac{1}{i\hbar}\Delta t H}|\psi\rangle$. Change is continuous in time, reversible, deterministic, linear in the wave function.

- Projective measurements: $|\psi\rangle \mapsto P_j|\psi\rangle/p_j$ with probability $p_j = \langle\psi|P_j|\psi\rangle$. Change is discontinuous in time, irreversible, non-deterministic, non-linear in the wave function.

Given that these are completely different, quantum physicists take great care to very carefully explain when to use the one and when to use the other. ... Huh huh, just kidding. Try to find a definition of "measurement" in your introductory textbook. So, what's up with that?

The standard presentation of quantum mechanics divides the world into a "quantum part" and a "classical part". The measurement rules connects the two. But it is not clear which degrees of freedom belong to which side of this *cut*.

Example: In the standard treatment of the Stern-Gerlach experiment, the spin is modeled quantum mechanically, but the spatial position of the atom classically. The spin-dependent movement of the atom is treated as a measurement. But it also seems reasonable to put the atom's position to the quantum side of the cut (Fig. 0.1). The interaction between spin and spatial coordinates is then described by a Hamiltonian time evolution. A measurement only takes place when an observer records the atom's position.

We can now state to aspects of quantum mechanic's *measurement problem*:

- *The pragmatic problem*: Why can physicists get away with being so vague about the notion of "measurement"? Why don't different modeling decisions produce different predictions? (We'll be able to answer this).

- *The philosophical problem*: Given that quantum mechanics is supposedly more fundamental than classical theories, how do we deal with the fact that its predictions are stated with respect to a classical world? Who's measuring the wave function of the universe? (We won't make progress here. In fact, there's no agreement what's the best solution to this issue. Or whether there is a solution. Or whether there was a problem in the first place. It's a mess.)

0.1.2 A quantum model for Stern-Gerlach

Hamiltonian for particle with spin interacting with an external magnetic field:

$$H = \frac{P^2}{2m} - \frac{\gamma\hbar}{2}\vec{B}\cdot\vec{\sigma}$$

Assume that $\vec{B} = Bz\vec{e}_z$. Then only the z-coordinate participates in the interaction, so nothing is lost by only treating the spin and the z-coordinate quantum-mechanically. The calculation is best done in interaction picture:

$$|\psi_I(t)\rangle = e^{-\frac{1}{i\hbar}tH_0}|\psi_S(t)\rangle = e^{\frac{1}{i\hbar}tH_I}|\psi_S(0)\rangle$$

describes the change of dynamics caused by an interaction term H_I, where

$$|\psi_S(t)\rangle = e^{\frac{1}{i\hbar}t(H_0+H_I)}|\psi_S(0)\rangle$$

is the usual Schrödinger-picture wave function. We will, of course, choose

$$H = H_0 + H_I, \qquad H_0 = \frac{P_z^2}{2m}, \qquad H_I = -\frac{\gamma\hbar B}{2}z\sigma_z.$$

First treat the case where the particle is initially in a momentum-0 eigenstate:

$$|\psi_S(t=0)\rangle = (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)|k=0\rangle.$$

Then, with $\delta := \frac{\hbar\gamma B}{2}$,

$$\begin{aligned}|\psi_I(t)\rangle &= e^{\frac{i\gamma}{2}tBz\sigma_z}|\psi_S(t=0)\rangle \\ &= \alpha\big(e^{\frac{i\gamma}{2}tBz}|\uparrow\rangle|k=0\rangle\big) + \beta\big(e^{-\frac{i\gamma}{2}tBz}|\downarrow\rangle|k=0\rangle\big) \\ &= \alpha|\uparrow\rangle|\delta t\rangle + \beta|\downarrow\rangle|-\delta t\rangle. \end{aligned} \tag{0.1}$$

This is an entangled state! A measurement of spin and momentum gives correlated outcomes:

$$\Pr[s, k+\mathrm{d}k] = \begin{cases} |\alpha|^2 & (s,k) = (\uparrow, +\delta t) \\ |\beta|^2 & (s,k) = (\downarrow, -\delta t) \end{cases}.$$

The marginal distribution for the spin variable alone is

$$\Pr[s] = \begin{cases} |\alpha|^2 & s = \uparrow \\ |\beta|^2 & s = \downarrow \end{cases},$$

exactly what we would have obtained by treating just the spin quantum mechanically. Thus: Using a quantum model for the spatial z-component does *not* change the prediction about the measured spin state. All it does is to entangle the measured and the measuring degree of freedom so that the global state becomes a superposition of consistent configurations. We could have included further degrees of freedom – e.g. the experimentalist observing the particle momentum. If we model them – simplifying slightly – as a two-dimensional system with (mental) states $|☺\rangle$ when seeing an upwards moving atom, and $|☹\rangle$ when encountering one moving downwards, a similar calculation would have resulted in

$$|\psi_I(t)\rangle = \alpha|\uparrow\rangle|\delta t\rangle|☺\rangle + \beta|\downarrow\rangle|-\delta t\rangle|☹\rangle.$$

Instead of a momentum eigenstate, let's use a more realistic Gaussian initial state. Write $|\psi_{k_0}\rangle$ for a Gaussian wave packet centered around k_0 in momentum space:

$$\langle k|\psi_{k_0}\rangle = (2\pi)^{-1/4} e^{-\frac{(k-k_0)^2}{4}}.$$

Then

$$e^{\frac{i\gamma}{2}Bz}|\psi_{k_0}\rangle = |\psi_{k_0+\delta t}\rangle$$

so that, if we take $|\psi_S(0)\rangle = |\psi_0\rangle$,

$$|\psi_I(t)\rangle = \alpha|\uparrow\rangle|\psi_{\delta t}\rangle + \beta|\downarrow\rangle|\psi_{-\delta t}\rangle.$$

We can now see that the correlations between spin and position now build up over time. Indeed:

$$\Pr[s, k + \mathrm{d}k] = \frac{1}{\sqrt{2\pi}} \begin{cases} |\alpha|^2 e^{-\frac{(k-\delta t)^2}{2}} \mathrm{d}k & s = \uparrow \\ |\beta|^2 e^{-\frac{(k+\delta t)^2}{2}} \mathrm{d}k & s = \downarrow \end{cases}.$$

At $t = 0$, the momentum distribution is independent of the spin state. For times $t \simeq 1/\delta$, the two spin-dependent Gaussian distributions become distinct, but overlap significantly. Only for $t \gg 1/\delta$ does the sign of a measured momentum value identify the spin state with certainty.

It is instructive to compute the reduced density matrix for the spin. From

$$|\psi_I(t)\rangle\langle\psi_I(t)| = |\alpha|^2 \, |\uparrow\rangle\langle\uparrow| \otimes |\psi_{\delta t}\rangle\langle\psi_{\delta t}| + \alpha\beta^* \, |\uparrow\rangle\langle\downarrow| \otimes |\psi_{\delta t}\rangle\langle\psi_{-\delta t}| + \dots$$

and

$$\mathrm{tr}\,|\psi_{\delta t}\rangle\langle\psi_{-\delta t}| = \langle\psi_{-\delta t}|\psi_{\delta t}\rangle = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{(-\delta t-k)^2+(\delta t-k)^2}{4}} \mathrm{d}k = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{k^2+(\delta t)^2}{2}} \mathrm{d}k = e^{-(\delta t)^2/2},$$

we can read off the reduced density matrix in $\{|\uparrow\rangle, |\downarrow\rangle\}$-basis:

$$\rho_{\text{spin}}(t) = \mathrm{tr}_{\text{space}}\, |\psi_I(t)\rangle\langle\psi_I(t)| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-(\delta t)^2} \\ \alpha^*\beta e^{-(\delta t)^2} & |\beta|^2 \end{pmatrix}.$$

Thus, the state of the spin part alone *dephases* from a pure state at $t = 0$ to a probabilistic mixture of $|\uparrow\rangle$ and $|\downarrow\rangle$ for times $t \gg 1/\delta$. The entropy (of entanglement) gradually builds up from $S(t = 0) = 0$ to

$$S(t \to \infty) = -|\alpha|^2 \log|\alpha|^2 - |\beta|^2 \log|\beta|^2.$$

In fact, these simple calculations offer much more than a solution to the "pragmatic aspect" of the measurement problem. Here's what we can take away:

- Q.: *Are measurements discontinuous in time?*
  A.: Nope! Correlations between the measured system and the environment are built up at a time scale proportional to the inverse coupling strength. The instantaneous process postulated in introductory QM can be understood as an effective description valid for times much larger than that.

- Q.: *Are measurements irreversible?*
  A.: The dynamics on the quantum side of the cut is reversible in theory – the final measurement still isn't. This doesn't lead to practical contradictions, though. Assume we put the hole universe, except for ourselves, on the quantum side. What would it take to reverse the measurement after a blob of silver (in the Stern-Gerlach case) has been deposited on a plate, but before we have looked at it? The deposit will have interacted with an enormous number of degrees of freedom: phonons in the plate, the cosmic background radiation, thermal photons that have since zoomed off into the sky at the speed of light. Clearly, *for all practical purposes* ("FAP"), it is impossible to reverse those interactions. Thus, once a macroscopic record of an event exists, the irreversibility introduced by QM's measurement postulate does not change anything FAP. *Philosophically*, it might still be a thorny issue! This is all good news if you like to compute things (no immediate contradiction). It's bad news if you like to understand foundational questions, because there seems little empirical guidance on offer for how to handle this conceptual inconsistency.

- Q.: *How does a classical world appear in quantum theory*[1]*?*
  A.: When's the last time you've come across a momentum eigenstate in the real world? Particles do seem to be "here *or* there", not "here *and* there" in the sense of quantum superpositions. But why is that, given the unitary invariance of QM's state space, which treats all bases equivalently? Well, consider the spin degree of freedom of our model. On short time scales, an arbitrary initial state will dephase into a probabilistic mixture of spin-up and spin-down states with respect to the z-axis. After the dephasing time, an unrelated observer will therefore find the spin in a σ_z-eigenstate and will not encounter superpositions. Recall what distinguishes the z-basis: *It is the one in which the interaction takes place*! While the *kinematics* (i.e. the set of all states) of QM are unitarily invariant, the *dynamics* are not. The bases which we perceive as "classical" are dynamically selected, and the creation of probabilistic mixtures is a result of entanglement building up. This process is called *decoherence*. Interactions are *local*, which is why quantum systems usually appear to be well-localized in space. However, some interactions select for different bases: e.g. electrons bound in an atom couple to the environment via the electromagnetic field. This interaction involves the atomic energy basis. Therefore, the semi-classical description of electrons in terms of atomic quantum numbers ("n, l, m") makes sense.

- Q.: *In thermodynamics, there's tension between the fact that entropy increases, while microscopic dynamics is reversible. The buildup of entanglement seems like an elegant solution: Local randomness is created from globally reversible dynamics. Maybe all entropy is entanglement entropy. Is that a good way to think about the apparent increase of entropy?*
  A.: You betcha!

- Q.: *More prosaically, how does entanglement emerge in natural systems?*
  A.: When the Hamiltonian contains a coupling term between different degrees of freedom (and neither system is in an eigenstate of the coupling).

[1] That's a reference to the title of an influential book on the topic, co-authored by Cologne's very own Claus Kiefer.