Quantenmechanik

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Sheet 4 **Due: 30.04 um 12 Uhr**

1 Matrix mechanics (5 P)

In the old times there were two different models of quantum mechanics: the Matrix Mechanics, introduced in 1925 by Heisenberg and Born, and the Wave Mechanics, introduced in 1926 by Schrödinger. The dispute between them didn't last long, though, as already in 1926 Schrödinger suspected that they were equivalent, which was proven by von Neumann in 1929.

In this exercise we shall develop the Matrix Mechanics and take a look at the equivalence.

a) **(0,5 P)** Show that the quantum state $|\psi(t)\rangle = e^{-itH/\hbar}|\psi(0)\rangle$ is a solution of the Schrödinger equation $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle$.

b) **(0,5 P)** Show that the expectation value of an arbitrary observable A with the quantum state $|\psi(t)\rangle$ from item **1a** is equal to the expectation value of the observable $A^H = e^{itH/\hbar}Ae^{-itH/\hbar}$ with the quantum state $|\psi(0)\rangle$. A^H is the observable A in the Heisenberg picture.

c) **(1,5 P)** Let $H = -\gamma\hbar\sigma_z$ be the Hamiltonian of a Spin-$\frac{1}{2}$ particle in an homogeneous magnetic field in the z direction. Compute σ_x^H, the observable σ_x in the Heisenberg picture, and compute with that the time-dependent expectation value of the initial states $|\uparrow\rangle$ and $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ in this observable.

Reminder: Item **2a** from Sheet 2 can be helpful.

d) **(1 P)** Show that $[U(t), H] = 0$, where $U(t) = e^{-itH/\hbar}$ is the time evolution operator, and $[A, B] = AB - BA$ the commutator. With that, show that $H^H = H$, that is, that the Hamiltonian is the same in the Heisenberg and Schrödinger pictures.

e) **(1,5 P)** Show that the observable A^H from item **1b** is a solution of the Heisenberg equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}A^H = [A^H, H].$$

2 Fourier transform (5 P)

In the next lectures we shall see that the Fourier transform plays an important role in quantum mechanics, for example in the description of momentum. Therefore we will consider here a couple of properties of the Fourier transform. Their physical meaning will become clear in the next lectures.

In what follows $\psi : \mathbb{R} \to \mathbb{C}$ is a function that goes to 0 in the infinite limit: $\lim_{x\to\infty}\psi(x) = 0$. The Fourier transform is given by

$$\mathcal{F}[\psi](k) = \widetilde{\psi}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}x\, e^{-ikx}\psi(x)\,,$$

with the inverse transform

$$\mathcal{F}^{-1}[\widetilde{\psi}](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}k\, e^{ikx}\widetilde{\psi}(k) = \psi(x).$$

a) **(1 P)** Show that:

$$\psi(x) = \psi^*(x) \iff \widetilde{\psi}^*(k) = \widetilde{\psi}(-k),$$

where the star means complex conjugation.

b) **(2 P)** We define the translation operator T_a and the multiplication operator X as:

$$(T_a\psi)(x) = \psi(x-a), \qquad (X\psi)(x) = x\psi(x).$$

The derivative of ψ is written as ψ'. Show that

$$\begin{aligned} \mathcal{F}[T_a\psi](k) &= e^{-ika}\widetilde{\psi}(k)\,, \\ \mathcal{F}[\psi'](k) &= i(X\widetilde{\psi})(k)\,, \\ \mathcal{F}[X\psi](k) &= i\widetilde{\psi}'(k)\,. \end{aligned}$$

Hint: Partial integration. Inserting $\psi = \mathcal{F}^{-1}[\widetilde{\psi}]$ could help.

c) **(2 P)** Let $\psi(x) = e^{-x^2}$. The integral that defines $\widetilde{\psi}$ is difficult to solve directly. We shall do it here indirectly. Show that (with partial integration) the relation

$$\frac{\partial}{\partial k}\widetilde{\psi}(k) = -\frac{k}{2}\widetilde{\psi}(k).$$

Solve this differential equation. The solution is unique up to a normalisation constant, that does not need to be computed here.