Quantenmechanik

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Sheet 7 **Due: 21.05 um 12 Uhr**

1 Our favourite child, the harmonic oscillator (10 P)

The Hamiltonian of the harmonic oscillator is given by

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2,$$

and when written in terms of the creation and annihilation operators $a^\dagger$ and a, it becomes

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\mathbb{1}\right),$$

where

$$a = \frac{1}{\sqrt{2}}\left(\frac{X}{x_0} + i\frac{P}{p_0}\right) \quad \text{und} \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{X}{x_0} - i\frac{P}{p_0}\right)$$

for $x_0 = \sqrt{\hbar/m\omega}$ and $p_0 = \sqrt{\hbar m\omega}$.

To work with these operators, the commutation relations $[a, a^\dagger] = \mathbb{1}$, $[a^\dagger, a^\dagger] = [a, a] = 0$ are useful, and the effect of a and $a^\dagger$ on the eigenstates of $a^\dagger a$ is given by

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad \text{and} \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

a) **(1 P)** The *coherent states* $|\alpha\rangle$ are defined as eigenstates of the annihilation operator a with eigenvalue α, that is, $a|\alpha\rangle = \alpha|\alpha\rangle$. We want to write them down in terms of the eigenstates of the harmonic oscillator $|n\rangle$, as

$$|\alpha\rangle = C\sum_{n=0}^{\infty} f_n(\alpha)|n\rangle.$$

Determine the coefficients $f_n(\alpha)$ and the normalisation factor C. Test whether the coherent states $|\alpha\rangle$ and $|\beta\rangle$ for $\alpha \neq \beta$ are orthogonal by computing $|\langle\beta|\alpha\rangle|^2$.

b) **(1 P)** Show that the time evolution of the coherent states is given by

$$|\psi(t)\rangle = e^{-\frac{1}{2}i\omega t}e^{-\frac{|\alpha_0|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}e^{-i\omega n t}|n\rangle = e^{-\frac{1}{2}i\omega t}|\alpha(t)\rangle,$$

where $|\alpha(t)\rangle$ is a coherent state with eigenvalue $\alpha(t) = \alpha_0 e^{-i\omega t}$.

c) **(2 P)** Compute the *time-dependent* expectation values of the position and momentum operators for the coherent state $|\alpha(t)\rangle$. With that, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\alpha(t)|P|\alpha(t)\rangle = -\langle\alpha(t)|\frac{\mathrm{d}}{\mathrm{d}x}V(X)|\alpha(t)\rangle,$$

for $V(x) = \frac{1}{2}m\omega^2 x^2$. What is the meaning of this equation?

Hint: Compute first $a|\alpha(t)\rangle$, and remember that $(A|\psi\rangle)^\dagger = \langle\psi|A^\dagger$. This equation is the famous Ehrenfest theorem, that will be spoken about in the lecture.

d) **(2 P)** Compute the product of variances $\Delta X \Delta P$ for an arbitrary coherent state $|\alpha\rangle$.

Hint: If you computed it correctly, you should obtain the surprising result that the product of variances does not depend on α.

e) **(2 P)** Consider the *displaced* ground state of the harmonic oscillator $|0_L\rangle$, that in the position representation is given by

$$\langle x|0_L\rangle = \frac{1}{\sqrt{x_0\sqrt{\pi}}} e^{-\frac{1}{2}\left(\frac{x-L}{x_0}\right)^2}.$$

Write $|0_L\rangle$ in the $|n\rangle$ basis and compare that with the result of item **1a**.

Hint: The state $|0_L\rangle$ is the state $|0\rangle$ translated by L, that is, $|0_L\rangle = e^{-\frac{i}{\hbar}LP}|0\rangle$. Write P in terms of $a^\dagger$ and a and use the Baker-Campell-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]},$$

that is true for operators A, B whose commutator $[A,B]$ respect the condition $[A,[A,B]] = [B,[A,B]] = 0$.

f) **(1,5 P)** Since the time evolution of a coherent state is a coherent state, $|\alpha(t)\rangle$ must respect the differential equation

$$a|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle,$$

and with that we can compute its position representation $\langle x|\alpha(t)\rangle$. Solve then

$$\frac{1}{\sqrt{2}}\left(\frac{X}{x_0} + i\frac{P}{p_0}\right)\langle x|\alpha(t)\rangle = \alpha(t)\langle x|\alpha(t)\rangle$$

and normalise the solution to find that

$$\langle x|\alpha(t)\rangle = \sqrt[4]{\frac{p_0}{\pi\hbar x_0}} e^{-\frac{x_0 p_0}{\hbar}\Re(\alpha(t))^2} \exp\left(-\frac{p_0}{2x_0\hbar}x^2 + \frac{p_0\alpha(t)\sqrt{2}}{\hbar}x\right),$$

modulo an irrelevant global phase.

g) **(0,5 P)** The quantum state

$$|\mathrm{cat}(t)\rangle = \frac{1}{\sqrt{2(1+e^{-2|\alpha(t)|^2})}}\left(|\alpha(t)\rangle + |-\alpha(t)\rangle\right)$$

is known as Schrödinger cat state. Sketch in 3D the absolute value squared of $|\mathrm{cat}(t)\rangle$ as a function of x and t.

2 (Bonus exercise) The Wigner quasiprobability distribution (2 P)

The Wigner quasiprobability distribution is a representation of wavefunctions in the usual $x-p$ phase space. It is quite useful for studying the quantum-classical correspondence. The Wigner representation of a wavefunction $\psi(x)$ is defined as

$$W(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}y\; \psi^*(x+y)\psi(x-y)e^{2ipy/\hbar}$$

Sketch in 3D the Wigner representation of $\langle x|\mathrm{cat}(t)\rangle$ für $t=0$. **Attention:** $W(x,p)$ is always a real number.