Classical Mechanics

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Exercise sheet 10 **Due: December, 21 at 12:00**

1 Phase space flow around an unstable equilibrium

The solutions to the Hamilton equations form trajectories, i.e., curves in phase space. For each point in phase space there is a unique curve passing through it (for time-independent Hamilton functions). In other words, the trajectories never cross each other in phase space. In this exercise we are going to investigate this for a specific example.

Consider a particle with mass m, moving along a straight line, in the potential $V(x) = -\frac{\alpha}{2}x^2$, where $\alpha > 0$[1]. The Hamilton function of the particle is

$$H(x,p) = \frac{1}{2m}p^2 - \frac{\alpha}{2}x^2. \tag{1}$$

This system has only one equilibrium solution, namely that the particle is at rest at $x = 0$, which is an unstable equilibrium. In terms of phase space, this corresponds to the 'trajectory' $\big(x(t), p(t)\big) = (0,0)$. In other words, the equilibrium solution corresponds to a single point in phase space.

a) *Make a sketch of the flow in phase space around the equilibrium point by determining and drawing the energy level sets (especially $E = 0$) and the direction of the flow on these curves.* You do not have to draw a very exact picture, a rough sketch is sufficient (alternatively you can plot it with some software.)

Hint: The energy level set corresponding to energy E is the set of points (x,p) such that $H(x,p) = E$. Since energy is conserved for time-independent Hamilton functions, it follows that the flow in phase space moves along these energy level sets. In the present case, where phase space is two-dimensional, the level set would typically consist of curves. Since $(\dot{x},\dot{p})$ gives the direction of the motion in phase space, it follows that we obtain the direction of the flow along the energy level curves from the Hamilton equations[2]. **(2 points)**

b) In the second part of problem a) you will find curves of constant energy (with the same energy as the equilibrium) on which the flow is directed towards as well as away from the equilibrium. At first sight it may thus look like there is a lot of crossings of solutions going on at the equilibrium point $(0,0)$. This is a bit worrying, so we need to take a closer look at what is happening.

Find the complete solutions to the Hamilton equations corresponding to the Hamilton function (1).

(2 points)

c) *Determine all solutions that have the same energy as the equilibrium solution.* **(2 points)**

d) *For the solutions in c) that correspond to motion towards the equilibrium (but not initially being at equilibrium), would the particle ever reach the equilibrium in any finite time? With this insight, explain why the findings in a) is not in contradiction with the statement that the solutions cannot cross in phase space.* **(2 points)**

[1] This is sometimes referred to as the inverted Harmonic oscillator. It is a quadratic potential just as the Harmonic oscillator, but where the potential is, so to speak, turned upside down.

[2] In higher dimensions the energy level sets would typically consist of surfaces, and the flow would stay within these surfaces.

2 Poisson brackets

For two functions $f(q_1, \ldots, q_N, p_1, \ldots, p_N, t)$ and $g(q_1, \ldots, q_N, p_1, \ldots, p_N, t)$, the Poisson bracket is defined as[3]

$$\{f, g\} = \sum_{n=1}^{N} \left(\frac{\partial f}{\partial p_n} \frac{\partial g}{\partial q_n} - \frac{\partial f}{\partial q_n} \frac{\partial g}{\partial p_n} \right). \tag{2}$$

The Poisson brackets satisfy various convenient rules, such as

$$\{q_k, q_l\} = 0, \quad \{p_k, p_l\} = 0, \quad \{p_k, q_l\} = \delta_{kl},$$
$$\{f, g\} = -\{g, f\}, \quad \{f + g, h\} = \{f, h\} + \{g, h\}, \quad \{fg, h\} = f\{g, h\} + \{f, h\}g.$$

When calculating with Poisson brackets it is often enough to use of these relations, without having to think about the definition (2), as we shall see in this exercise.

a) *Show that*

$$\{q_j, p_k^n\} = -np_k^{n-1}\delta_{jk}, \quad \{p_j, q_k^n\} = nq_k^{n-1}\delta_{jk}, \quad n = 1, 2, 3, \ldots$$

Hint: This is efficiently shown via an induction proof. **(2 points)**

b) The angular momentum of a particle with position $\vec{q}$ and momentum $\vec{p}$, is given by $\vec{L} = \vec{q} \times \vec{p}$. For $\vec{L} = (L_1, L_2, L_3)$, one can write $L_j = \sum_{kl} \epsilon_{jkl} q_k p_l$, where ϵ_{jkl} is the Levi-Civita symbol[4].

Show that

$$\{q_n, L_j\} = -\sum_k \epsilon_{njk} q_k, \quad \{p_n, L_j\} = -\sum_l \epsilon_{njl} p_l, \quad j, n = 1, 2, 3, \tag{3}$$

and

$$\{\vec{q}^2, L_j\} = 0, \quad \{\vec{p}^2, L_j\} = 0, \quad j = 1, 2, 3. \tag{4}$$

Hint: Note that $(\vec{a} \times \vec{b})_j = \sum_{kl} \epsilon_{jkl} a_k b_l$ and $\vec{a} \times \vec{a} = 0$. Note also that whenever you permute two indices in ϵ_{jkl}, then it changes sign, e.g., $\epsilon_{jkl} = -\epsilon_{kjl} = \epsilon_{klj}$.

(4 points)

[3] The definition of the Poisson bracket occurs in two variations that differ in the choice of the overall sign. This choice affects the sign of the bracket $\{p_k, q_l\}$.

[4] The Levi-Civita symbol in three dimensions is $\epsilon_{jkl} = \begin{cases} 1 & (jkl) = (123), (312), (231) \\ -1 & (jkl) = (213), (321), (132) \\ 0 & \text{else} \end{cases}$.

3 Conserved quantities via Poisson brackets

Suppose that we have two particles of mass m that move in three-dimensional space, and interact with each other via a quadratic potential. This can be described with the following Hamilton function

$$H(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2) = \frac{1}{2m}\vec{p}_1^{\,2} + \frac{1}{2m}\vec{p}_2^{\,2} - \alpha\|\vec{q}_1 - \vec{q}_2\|^2.$$

Let $\vec{L}^{(1)} = \vec{q}_1 \times \vec{p}_1$ and $\vec{L}^{(2)} = \vec{q}_2 \times \vec{p}_2$ be the angular momentum vectors of particle 1 and 2, respectively.

a) *Show that*

$$\left\{H, L_j^{(1)} + L_j^{(2)}\right\} = 0, \quad j = 1, 2, 3. \tag{5}$$

Hint: There are various observations that make the derivation quicker. For example, $\{f(\vec{q}_1, \vec{p}_1) + f(\vec{q}_2, \vec{p}_2), g(\vec{q}_1, \vec{p}_1) + g(\vec{q}_2, \vec{p}_2)\} = \{f(\vec{q}_1, \vec{p}_1), g(\vec{q}_1, \vec{p}_1)\} + \{f(\vec{q}_2, \vec{p}_2), g(\vec{q}_2, \vec{p}_2)\}$. The relations (3) and (4) can be useful.

(4 points)

b) *In more physical terms, what is it that you have proved with equation (5)?* **(2 points)**