Classical Mechanics

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Exercise sheet 11 **Due: January, 11 at 12:00**

A function $F = F(\vec{r}, \vec{p})$ on phase space is a *conserved quantity*, with respect to the Hamilton function H, if $\{H, F\} = 0$. A collection of conserved quantities $\{F_j\}_{j=1,\dots,N}$ are called *independent* if

$$\{F_j, F_k\} = 0, \quad j, k = 1, \dots, N.$$

If a system possesses a constant of motion it can be used to simplify the equations of motion. In general, N independent conserved quantities reduce the number of degrees of freedom from f to $f - N$. Concretely, this can be done by using the equations $F_j = \text{const}$ to eliminate coordinates. A system is called *integrable* if it has as many independent conserved quantities as it has degrees of freedom, i.e., $f = N$. In this case, the solution to the equations of motion can be directly found by integrating conservation laws.

1 Conserved quantities of centrally symmetric Hamiltonians

a) Consider a particle with mass m in a centrally symmetric potential V. This has the Hamilton function

$$H(\vec{r}, \vec{p}) = \frac{\vec{p}^2}{2m} + V(\|\vec{r}\|). \tag{1}$$

Show that the angular momentum vector $\vec{L}$ and $\vec{L}^2$ are conserved quantities.

Hint: Recall exercise 2b) on sheet 10. For the potential, it might be easiest to directly use the definition of the Poisson bracket $\{f, g\} = \sum_{n=1}^{N} \left(\frac{\partial f}{\partial p_n} \frac{\partial g}{\partial r_n} - \frac{\partial f}{\partial r_n} \frac{\partial g}{\partial p_n} \right)$ and the chain rule. Recall also that $L_j = \sum_{kl} \epsilon_{jkl} r_k p_l$.

(3 points)

b) *Using a), how many conserved quantities can you name? Which of them are independent? Is the system integrable?* You are allowed to use the relation $\{L_j, L_k\} = -\sum_l \epsilon_{jkl} L_l$ without proof. **(3 points)**

2 Conserved quantities simplify the equations of motion

To illustrate how conserved quantities actually simplify the analysis, we shall here partially solve the equations of motion corresponding to the Hamilton function (1). The idea is that we should rewrite the Hamiltonian in terms of conserved quantities.

We will make use of a coordinate transformation to spherical coordinates, i.e.

$$\vec{r} = r \begin{bmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{bmatrix} \equiv r\hat{r}.$$

To simplify the calculations, you can use the following relations that will come in handy. The Cartesian unit vectors $\hat{x}, \hat{y}, \hat{z}$ and the spherical unit vectors $\hat{r}, \hat{\theta}, \hat{\varphi}$ are related by:

$$\begin{aligned} \hat{r} &= \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}, \\ \hat{\theta} &= \cos\theta\cos\varphi\hat{x} + \cos\theta\sin\varphi\hat{y} - \sin\theta\hat{z}, \\ \hat{\varphi} &= -\sin\varphi\hat{x} + \cos\varphi\hat{y}. \end{aligned}$$

Furthermore, $\hat{r}, \hat{\theta}, \hat{\varphi}$ are orthogonal and fulfill

$$\hat{r} \times \hat{\theta} = \hat{\varphi}, \qquad \hat{r} \times \hat{\varphi} = -\hat{\theta}, \qquad \hat{\theta} \times \hat{\varphi} = \hat{r}.$$

a) *Write the Hamilton function (1) in spherical coordinates and determine the canonical momenta in terms of r, θ, φ and their derivatives.*

Hint: It might be easier to first transform the Hamilton function to its corresponding Lagrangian, make the change of variables to spherical coordinates in the Lagrangian, and then transform back again to the Hamilton-function. It can be useful to first show that $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\hat{\varphi}$. Use exercise 2 on sheet 9 for the Legendre transform.

(4 points)

b) *Express $\vec{L}^2$ in spherical coordinates, and relate it to p_θ and p_φ.*

Hint: Use $\vec{L} = m\vec{r} \times \dot{\vec{r}}$ and $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\hat{\varphi}$ from a).

(4 points)

c) *Compute L_3 in spherical coordinates and express it using p_θ and p_φ.*

Hint: With $\vec{L}$ expressed in terms of the spherical unit vectors, one can compute $L_3 = \hat{z} \cdot \vec{L}$.

(2 points)

d) *Use the conserved quantities in order to show that for constant angular momentum $\vec{L}^2 = \ell^2$, the Hamiltonian in spherical coordinates can be written as*

$$\tilde{H}(r, \theta, \varphi, p_r, p_\theta, p_\varphi) = \frac{1}{2m}p_r^2 + \frac{\ell^2}{2mr^2} + V(r). \tag{2}$$

(2 points)

e) By (2) we have reduced the the original problem to a single degree of freedom, i.e., the radial motion. One could now proceed to determine the solutions, e.g. from energy conservation, but we have already discussed that. Instead we explore the phase-space flow of (2), for the special case of the Kepler problem

$$V(r) = -\frac{k}{r},$$

for some constant k.

Use some plotting software to draw the energy level curves of (2).

Hint: It may not be entirely trivial to make a plot where one actually see the bound orbits $E < 0$. It can be convenient to first change variables to $\tilde{r}$, $\tilde{p}_r$, and $\tilde{E}$, such that $r = \tilde{r}\frac{\ell}{mk}$, $E = \tilde{E}\frac{k^2 m}{\ell}$, $p_r^2 = \tilde{p}_r^2\frac{m^2k^2}{\ell}$, which results in $\tilde{E} = \frac{1}{2}\tilde{p}_r^2 + \frac{1}{2\tilde{r}^2} - \frac{1}{\tilde{r}}$. By plotting energy level curves for $\tilde{E} = -0.5$ to $\tilde{E} = 0.5$ in steps of 0.05, in the region $0.2 \leq \tilde{r} \leq 10$ and $-1 \leq \tilde{p}_r \leq 1$, one should be able to see both bound and unbound orbits.

(2 points)

Remark: You might have wondered what happened to the conserved quantity L_3 and want to argue that there is something wrong in our approach, since it was needed for integrability. However, H only depends on $\vec{L}^2$ which, in turn, depends of course on L_3. But if one actually wants to compute the solutions using conservation laws, the relations for L_3 and $\vec{L}^2$ are both needed.