Classical Mechanics

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Exercise sheet 12 **Due: January, 18 at 12:00**

1 Solving the equations of motion via canonical transformations

In this exercise we shall use canonical transformations in order to solve the equations of motion.

a) Consider the transformations from (q, p) to (Q, P) defined by

$$Q = \alpha p q^{\gamma}, \quad P = \beta q^{\delta}, \tag{1}$$

where α, β, γ and δ are constants. *What are the conditions on α, β, γ and δ for (1) to be a canonical transformation?*

Hint: Recall the characterization of canonical transformations in terms of Poisson brackets, and the definition of the Poisson bracket.

(3 points)

b) *Find a canonical transformation from (q, p) to (Q, P) that transforms*

$$H(q, p) = \frac{1}{2}p^2q^4 + \frac{1}{2q^2} \tag{2}$$

into the Hamilton function of the Harmonic oscillator

$$H(Q, P) = \frac{1}{2}P^2 + \frac{1}{2}Q^2.$$

Hint: There was a reason for why we bothered to do exercise a). **(2 points)**

c) Use the result in b) in order to derive the solutions to Hamilton's equations corresponding to (2). **(2 points)**

2 Liouville's theorem

Consider a free particle with mass m that moves along a straight line, i.e., the particle has the Hamilton function

$$H(x, p) = \frac{1}{2m}p^2.$$

For $a, b > 0$ consider the rectangle R in phase space, defined by $-a \leq x \leq a$, $-b \leq p \leq b$.

a) If we regard R as a set of possible initial conditions for the particle at time $t = 0$, then this set of points will, for every time $t \geq 0$, evolve to a new set $R(t)$.

Determine $R(t)$ and sketch the shape. **(3 points)**

b) *Determine the area of $R(t)$ and compare with R.* **(1 points)**

3 Generating functions for canonical transformations

Consider a function $F_1(q, Q)$ of the old coordinates q and the new coordinates Q. This function does implicitly define a canonical transformation between (q, p) and (Q, P) via the two equations

$$p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}. \tag{3}$$

The function F_1 is referred to as the generating function of the transformation.

a) Consider the function

$$F_1(q, Q) = \frac{m\omega}{2} q^2 \frac{1}{\tan Q},$$

where m and ω are some constants. *Use the relations (3) in order to express q and p as functions of Q and P.* **(2 points)**

Hint: The relations (3) gives p and P as functions of q and Q, and you have to transform these so that you obtain q and p as functions of Q and P. Do not worry about whether the square roots are well defined, or the sign of the roots.

b) Consider the Hamilton function for the harmonic oscillator

$$H(q, p) = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} q^2.$$

Express H in terms of the new variables Q and P. What is the solution of the corresponding equations of motion? **(2 points)**

Remark: One could transform the solutions back to the original (q, p) and thus obtain the solutions to the equations of motion of the harmonic oscillator.

4 Generating functions for canonical transformations again

In the previous exercise we considered generating functions $F_1(q, Q)$. However, one can also use generating functions $F_2(q, P)$, $F_3(p, Q)$, or $F_4(p, P)$. As an example we are here going to consider generating functions $F_3(p, Q)$. Such functions define canonical transformations between (q, p) and (Q, P) if

$$q = -\frac{\partial F_3}{\partial p}, \quad P = -\frac{\partial F_3}{\partial Q}. \tag{4}$$

Note the difference in signs compared to (3)!

a) Consider the function

$$F_3(p, Q) = -(e^Q - 1)^2 \tan p.$$

By using (4), determine Q and P as functions of q and p.

Hint: As in the previous exercise, do not worry about square roots (or logarithms) being well defined, or which branches to take. **(2 points)**

b) *Confirm, by using Poisson brackets, that the functions $Q(q, p)$ and $P(q, p)$ obtained in a) define a canonical transformation from (q, p) to (Q, P).* **(3 points)**

5 Deriving the conditions for generating functions

This exercise gives no points, but gold stars! If you feel that it is unclear where the conditions (3) and (4) come from, then this is the exercise for you.

One method to derive conditions like (3) and (4) is based on the Hamiltonian version of the variational principle. Recall that the action functional can be written $\mathcal{S}[\vec{q}] = \int_{t_i}^{t_f} L(\vec{q}, \dot{\vec{q}}, t)dt$. By recalling the relation between the Lagrangian and the Hamilton function, one can rewrite this as

$$\mathcal{S}[\vec{q}, \vec{p}] = \int_{t_i}^{t_f} \sum_j \dot{q}_j p_j - H(\vec{q}, \vec{p}, t)dt. \tag{5}$$

(By extremizing this integral one obtains Hamilton's equations.)

Suppose now that we have a new set of coordinates and conjugate momenta $\vec{Q}, \vec{P}$, and a new Hamilton function $\tilde{H}(\vec{Q}, \vec{P}, t)$ thus leading to the new functional

$$\tilde{\mathcal{S}}[\vec{Q}, \vec{P}] = \int_{t_i}^{t_f} \sum_j \dot{Q}_j P_j - \tilde{H}(\vec{Q}, \vec{P}, t)dt. \tag{6}$$

If the integrands in (5) and (6) are identical up to a total time derivative $\frac{dF}{dt}$ of some function F[1], then the two functionals (5) and (6) yield equivalent equations of motion. In other words, if

$$\sum_j \dot{q}_j p_j - H(\vec{q}, \vec{p}, t) = \sum_j \dot{Q}_j P_j - \tilde{H}(\vec{Q}, \vec{P}, t) + \frac{dF}{dt}, \tag{7}$$

then $(\vec{q}, \vec{p}, H)$ describe the same physics as $(\vec{Q}, \vec{P}, \tilde{H})$[2].

a) Suppose that we let

$$F = F_1(\vec{q}, \vec{Q}, t),$$

for some function F_1. *Show that a sufficient condition for (7) to be valid is that*

$$p_j = \frac{\partial F_1}{\partial q_j}, \qquad P_j = -\frac{\partial F_1}{\partial Q_j}, \qquad \tilde{H}(\vec{Q}, \vec{P}, t) = H(\vec{q}, \vec{p}, t) + \frac{\partial F_1}{\partial t}. \tag{8}$$

(0 points, but a gold star!)

Comment: The condition (3) is a special case of (8).

b) Suppose that we instead let

$$F = \sum_j q_j p_j + F_3(\vec{p}, \vec{Q}, t),$$

for some function F_3. *Show that*

$$q_j = -\frac{\partial F_3}{\partial p_j}, \qquad P_j = -\frac{\partial F_3}{\partial Q_j}, \qquad \tilde{H}(\vec{Q}, \vec{P}, t) = H(\vec{q}, \vec{p}, t) + \frac{\partial F_3}{\partial t}. \tag{9}$$

is a sufficient condition for (7) to hold.

(0 points, but a gold star!)

Comment: The relations (9) yields (4) as a special case.

If one has the energy, one can also derive the conditions for F_2 and F_4, which are based on the assumptions

$$F = -\sum_j Q_j P_j + F_2(\vec{q}, \vec{P}, t), \qquad F = \sum_j q_j p_j - \sum_j Q_j P_j + F_4(\vec{p}, \vec{P}, t).$$

[1] F can be a function of $\vec{q}$, $\vec{p}$, $\vec{Q}$, $\vec{P}$ and t, but not of $\dot{\vec{q}}$, $\dot{\vec{p}}$, $\dot{\vec{Q}}$, $\dot{\vec{P}}$.

[2] One way of thinking of this equation is that we regard all Q and P as being functions of q and p, or all q and p as being functions of Q and P.